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Conditional probability on MV-algebras

Tomáš Kroupa[a,b,*]

[a] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic
[b] Faculty of Electrical Engineering, Czech Technical University, Technická 2, 166 27 Praha 6, Dejvice, Czech Republic

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Abstract

An appropriate definition of a conditional probability on an MV-algebra was an open problem mentioned in Riečan and Mundici (Handbook of Measure Theory, North-Holland, Amsterdam, 2002). We propose some concept of conditional probability (state) on a σ-complete MV-algebra with product. Its basic properties will be proven and it will also be demonstrated that the conditional probability in classical probability theory is a special case of this definition. Moreover, the paper contains also a discussion of the interpretation of fuzzy sets and conditioning in fuzzy probability theory.

MSC: 60B99

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1. Introduction

Probability on MV-algebras has been evolving as a generalization of classical probability theory towards algebraic structures enabling to model and capture different kinds of uncertainty such as vagueness that is usually expressed by means of fuzzy set theory. An MV-algebra is a many-valued generalization of a Boolean algebra. MV-algebras can be viewed as algebraic generalizations of certain collections of fuzzy sets, so-called Łukasiewicz tribes, as well. Łukasiewicz tribes means for theory of MV-algebras the same as Boolean σ-algebras of sets means for theory of Boolean algebras; the relation between a σ-complete Boolean algebra and its set representation is clarified

* Corresponding author. Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic.
E-mail address: kroupa@utia.cas.cz (T. Kroupa).

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by the Loomis–Sikorski theorem [7]. A version of this theorem has also appeared [3,5] in theory of MV-algebras enabling hereby to represent each σ-complete MV-algebra with a Łukasiewicz tribe of fuzzy sets. The Loomis–Sikorski theorem for MV-algebras counts as an important tool of this paper.

Riečan and Mundici formulated in [6] an open problem concerning the reasonable definition of a conditional probability on an MV-algebra. This issue is the subject of this article. We propose a concept of a conditional probability which generalizes the classical (Kolmogorov) definition of a conditional probability on Boolean σ-algebra of sets.

The paper is structured as follows. Basic notions and results are repeated in Section 2, the new concept of a conditional probability (state) on an MV-algebra is introduced and further studied in Section 3. A possible interpretation of conditioning on an MV-algebra is briefly discussed in Section 4.

2. Basic notions

Let us summarize the basic results in MV-algebraic probability theory. See [2] for further details on MV-algebras and [6] for a thorough exposition on probability on MV-algebras.

2.1. *MV-algebras*

An MV-algebra is an algebra $\langle M, \oplus, \neg, \mathbf{0} \rangle$, where $\oplus$ is an associative and commutative binary operation on M having $\mathbf{0}$ as the neutral element, a unary operation $\neg$ is involutive with $a \oplus \neg\mathbf{0} = \neg\mathbf{0}$ for all $a \in M$, and, moreover, the identity $a \oplus \neg(a \oplus \neg b) = b \oplus \neg(b \oplus \neg a)$ is satisfied for all $a, b \in M$. Usually no ambiguity arises if we say that M is an MV-algebra.

The constant $\mathbf{1}$ and binary operations $\otimes, \ominus$ are defined as follows: $\mathbf{1} = \neg\mathbf{0}$, $a \otimes b = \neg(\neg a \oplus \neg b)$, $a \ominus b = a \otimes \neg b$. A partial order $\leqslant$ can be introduced on any MV-algebra by defining $a \leqslant b$ iff $\neg a \oplus b = \mathbf{1}$. An MV-algebra M is called *σ-complete* if every non-empty countable subset of M has a supremum in M. A sub-MV-algebra of M is a subset N of M containing the neutral element $\mathbf{0}$ of M, closed under the operations of M and endowed with the restriction of these operations to N.

Any MV-algebra M contains a Boolean algebra $\mathbf{B}(M)$ endowed with the natural restriction of operations inherited from M; namely, the *Boolean skeleton* $\mathbf{B}(M)$ is the set of all idempotent elements of M, i.e.

$$\mathbf{B}(M) = \{a \in M \mid a \oplus a = a\}.$$

We say that $\langle M, \oplus, \cdot, \neg, \mathbf{0} \rangle$ is an MV-algebra *with product* if $\langle M, \oplus, \neg, \mathbf{0} \rangle$ is an MV-algebra and a product $\cdot$ is a commutative and associative binary operation on M such that for all $a, b, c \in M$:

(1) $\mathbf{1} \cdot a = a$,
(2) $a \cdot (b \ominus c) = (a \cdot b) \ominus (a \cdot c)$.

It follows immediately that $\mathbf{0}$ acts as the zero element of M with respect to the product. In addition, the product is monotone in the following sense: if $a \leqslant b$, then $a \cdot c \leqslant b \cdot c$ for any $c \in M$.