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Checking the consistency of the solution in ordinal semi-democratic decision-making problems ☆

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ABSTRACT

An interesting decision-making problem is that aggregating multi-agent preference orderings into a consensus ordering, in the case the agents' importance is expressed in the form of a rank-ordering. Due to the specificity of the problem, the scientific literature encompasses a relatively small number of aggregation techniques. For the aggregation to be effective, it is important that the consensus ordering well reflects the input data, *i.e.*, the agents' preference orderings and importance rank-ordering.

The aim of this paper is introducing a new quantitative tool – represented by the so-called p indicators – which allows to check the degree of consistency between consensus ordering and input data, from several perspectives. This tool is independent from the aggregation technique in use and applicable to a wide variety of practical contexts, *e.g.*, problems in which preference orderings include omissions and/or incomparabilities between some alternatives. Also, the p indicators are simple, intuitive and practical for comparing the results obtained from different techniques. The description is supported by various application examples.

1. Introduction

A relatively little discussed decision-making problem is that of aggregating multi-agent preference orderings into a consensus ordering, in the specific case in which the agents' importance is expressed in the form of a rank-ordering. The problem of interest is characterized by the following elements:

- A set of alternatives to be prioritized (a, b, c, d, e, *etc.*).
- A set of m decision-making agents[1] (D_1, D_2, …, D_m) expressing their opinion on the alternatives, through preference orderings (*e.g.*, $a > [(b \sim c) \| d] > e > \ldots$, where symbols "$>$", "$\sim$" and "$\|$" respectively mean "strictly preferred to", "indifferent to" and "incomparable to").
- An importance hierarchy of the agents, which is expressed through a *linear* rank-ordering (*e.g.*, $D_1 > D_2 > (D_3 \sim D_4) > \ldots$) and not through a set of weights, as in most of the decision-making problems [19,25,26];
- A *consensus*[2] *ordering* of the alternatives, which represents the solution of the problem.

Franceschini et al. [13] classified this specific problem as *ordinal semi-democratic*; the adjective *semi-democratic* indicates that agents do not necessarily have the same importance, while *ordinal* indicates that their rank is defined by a linear ordering [20]. This problem is potentially adaptable to a large number of practical contexts, in which the agents' importance prioritization is dubious and controversial: in these situations, the formulation of a rank-ordering is certainly simpler and more intuitive than that of a set of weights defined on a *ratio* scale [5,14,17]. Possible examples are

- Management decision problems in which agents are the members of the management board of a company/organization and their importance reflects the relevant hierarchical level (*e.g.*, CEO, general manager(s), operations manager(s), office manager(s), *etc.*).
- Marketing decision problems in which agents are respondents to questionnaires/interviews and their importance reflects the relevant level of education (*e.g.*, Ph.D., M.Sc., B.Sc., high school, *etc.*);
- Competitions for academic positions in which agents are the members of committees and their importance reflects the relevant academic position (*e.g.*, full professor, associate professor, assistant professor, *etc.*).

The problem of aggregating preference orderings, when there is no agents' importance hierarchy (*fully democratic* case) or it is expressed through a set of weights, is quite old and has been studied in various fields, stimulating the development of a variety

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[1] By a "decision-making agent", we will consider any of a wide variety of different types of entities; examples could be human beings, individual criteria in a multicriteria decision-making process, intelligent entities in the field of artificial intelligence, etc.

[2] The adjective "consensus" means that this ordering should reflect the whole preference orderings as much as possible, even in the inevitable presence of divergences.

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of solution techniques [10,11,15]. For example, in the field of *social choice* and *voting theory*, we recall the pioneering method by Condorcet [6] and that by Borda [2], while, in the field of *multi-criteria decision making*, the Electre [9], Promethee [4,8], AHP [22] or TOPSIS [1] methods.

On the other hand, the *ordinal semi-democratic* problem has so far received relatively little attention, probably due to its specificity; we recall the contribution of Yager [24], proposing a practical aggregation technique (hereafter denominated as "Yager's algorithm", abbreviated as YA), and the contributions of Wang [27] and Franceschini et al. [13], presenting two ameliorative variants of the YA.

These aggregation techniques, and maybe those that will be proposed in the future, have their *pro* and *contra*. For this reason, an interesting question is: for a generic ordinal semi-democratic decision-making problem, how could we identify the best aggregation technique? We are aware that it is probably impossible to answer this question rigorously, since the "true" solution for a generic problem is not known *a priori* [9,7,29]. Nevertheless, the performance of different aggregation techniques maybe assessed, at least roughly, according to various aspects, such as

- The ability to produce a solution, which is consistent with the input data;
- The adaptability to a variety of input data, *e.g.*, preference orderings including omissions and/or incomparabilities between alternatives;
- The efficiency in using the input data for constructing the consensus ordering; *e.g.*, an algorithm that focuses on the lower/upper part of the preference orderings only or an algorithm that ignores the preference orderings of certain agents cannot be considered as very efficient.
- Computational complexity.

Among the aspects above, that concerning the consistency of the solution is particularly important. The argument of consistency has been used by Wang [27] and Franceschini et al. [13] to prove, at least at a conceptual level, the superiority of their variants with respect to the YA. In this context, *consistency* is defined as the ability of a solution to reflect the agents' preference orderings, while reflecting their importance hierarchy, *i.e.*, giving priority to the more important agents.

In the scientific literature, various tools for consistency checking have been proposed. A common feature is that they use some measures of correlation/similarity to compare the consensus ordering with the agents' preference orderings [21,28]. For example, popular statistics are the Kendall's tau, the Spearman's rho, Spearman's footrule, and Cayley's distance; see Ref. [18] for an overview. However, the application range of these tools may be limited by several aspects, such as

- The degree of "completeness" of the preference orderings; for example, many techniques are not easily applicable when some alternatives are tied, omitted or incomparable between each other [3].
- The form in which the importance hierarchy of the agents is expressed.

The aim of this paper is to provide a simple and practical tool to check the degree of consistency between the consensus ordering and the input data, for specific ordinal semi-democratic decision-making problems. The proposed tool enables two types of consistency evaluations

- At a *local* level, by comparing the consensus ordering with the preference ordering of each j-th agent.
- At a *global* level, by comparing the consensus ordering with the whole set of preference orderings, taking into account the agents' importance rank-ordering, under the assumption that the most important agents should have a predominant influence on the construction of the consensus ordering.

The consistency verification is performed through the so-called p indicators, as we will show later in the paper. The remainder of the paper is organized in two sections. Section 2 introduces the p indicators, focusing on their construction and practical use. The description is supported by several examples. Section 3 summarizes the original contributions of this research, focusing on its implications, limitations and possible future developments.

2. The p indicators

Before getting into the discussion of p indicators, we anticipate that they are virtually applicable to every aggregation technique, since they are obtained by comparing the paired-comparison relationships derived from one agent's preference orderings with those derived from the consensus ordering. The decision of using paired-comparison relationships is motivated by several reasons

1. They allow to express the preference between two alternatives in a natural and intuitive way.
2. They can be derived from (preference and consensus) orderings, even if some alternatives are tied, omitted or incomparable between each others. For the purpose of example, Fig. 1 illustrates the transformation of a fictitious *partial* ordering, with one omitted alternative (d) and two incomparable alternatives (a and e), into paired-comparison relationships [20].
3. They could also be derived from agents' judgements expressed in other forms (*e.g.*, measurements/evaluations on ordinal/interval/ratio scales), as long as they admit relationships of order among the alternatives.

The remainder of this section is divided into two sub-sections: Section 2.1 provides a general description of the p indicators, with an application example, and Section 2.2 illustrates the use of p indicators, for comparing the results provided by two different aggregation techniques, when applied to the same problem.

2.1. General description

Table 1 presents a summary scheme of the proposed indicators. The p_j indicators, associated with each j-th of the total m agents, allow to assess the consistency at a *local* level, while indicators p_A and p_B – in turn aggregated into p_O – allow to assess the consistency at a *global* level; the combination of all these indicators enables a structured evaluation of the degree of consistency between consensus ordering and input data, in a generic ordinal semi-democratic decision-making problem. These two types of indicators are defined and described in Sections 2.1.1 and 2.1.2, respectively.

2.1.1. p_j indicators

A preliminary operation for determining the p_j indicators is constructing a table, which contains the paired-comparison relationships obtained from the agents' preference orderings and the consensus ordering. For the purpose of example, let us consider the fictitious decision-making problem in Table 2, in which $m=4$ agents formulate their preference orderings concerning $n=5$ alternatives (a–e); the importance hierarchy of agents is expressed in the form of the rank-ordering $D_1 > D_2 > (D_3 \sim D_4)$. Incidentally, the preference ordering by D_4 is the same (partial) ordering