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Gröbner bases for polynomial systems with parameters

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ABSTRACT

Gröbner bases are the computational method par excellence for studying polynomial systems. In the case of parametric polynomial systems one has to determine the reduced Gröbner basis in dependence of the values of the parameters. In this article, we present the algorithm GröbnerCover which has as inputs a finite set of parametric polynomials, and outputs a finite partition of the parameter space into locally closed subsets together with polynomial data, from which the reduced Gröbner basis for a given parameter point can immediately be determined. The partition of the parameter space is intrinsic and particularly simple if the system is homogeneous.

Introduction

Let K be a field and $\overline{K}$ be an algebraically closed extension of K (e.g. $K = \mathbb{Q}$ and $\overline{K} = \mathbb{C}$). A parametric polynomial system over K is given by a finite set of polynomials $p_1, \ldots, p_r \in K[\overline{a}, \overline{x}]$ in the variables $\overline{x} = x_1, \ldots, x_n$ and parameters $\overline{a} = a_1, \ldots, a_m$, and one is interested in studying the solutions of the algebraic systems $\{p_1(a, \overline{x}), \ldots, p_r(a, \overline{x})\} \subset \overline{K}[\overline{x}]$ which are obtained by specializing the parameters to concrete values $a \in \overline{K}^m$.

The computational approach par excellence for studying algebraic systems is the method of Gröbner bases and several articles have already been dedicated to the application of the ideas of Gröbner bases in the parametric setting, e.g. (Gianni, 1987; Weispfenning, 1992; Becker, 1994; Kapur, 1995; Duval, 1995; Kalkbrenner, 1997; Van Hentenryck et al., 1997; Moreno-Maza, 1997; Dellière, 1999; González-López et al., 2000; Gómez-Díaz, 2000; Fortuna et al., 2001; Montes, 2002; O'Halloran and Schilmoeller, 2002; Gao and Wang, 2003; Weispfenning, 2003; Sato and Suzuki, 2003; Sato, 2005;

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González-Vega et al., 2005; Nabeshima, 2005; Manubens and Montes, 2006; Nabeshima, 2006; Suzuki and Sato, 2006; Wibmer, 2007; Chen et al., 2007; Inoue et al., 2007; Inoue and Sato, 2007; Manubens, 2008; Manubens and Montes, 2009).

The first very important step was the proof of the existence of a *Comprehensive Gröbner Basis* together with an algorithm to obtain one via *Gröbner systems* in Weispfenning (1992). These algorithms have been implemented by Schönfeld (1991), Pesch (1994) and Dolzmann et al. (2006). To explain this fundamental concept we fix a monomial order $\succ_{\overline{x}}$ on the variables and an ideal $I \subset K[\overline{a}][\overline{x}] = K[\overline{a}, \overline{x}]$ (with generating set $\{p_1, \dots, p_r\}$). For $a \in \overline{K}^m$ we denote by $I_a \subset \overline{K}[\overline{x}]$ the ideal generated by all $p(a, \overline{x}) \in \overline{K}[\overline{x}]$ for $p \in I$.

A *Gröbner system* for I and $\succ_{\overline{x}}$ is a finite set of pairs $\{(S_1, B_1), \dots, (S_s, B_s)\}$ such that

(i) The S_i's are locally closed subsets of $\overline{K}^m$ such that $\overline{K}^m = \cup S_i$.
(ii) The B_i's are finite subsets of $K[\overline{a}][\overline{x}]$ and $B_i(a) = \{p(a, \overline{x}) : p \in B_i\}$ is a Gröbner basis of I_a with respect to $\succ_{\overline{x}}$ for every $a \in S_i$.
(iii) For $p \in B_i$ the function $a \mapsto \text{lpp}(p(a, \overline{x}))$ is constant on S_i. In particular, $a \mapsto \text{lpp}(I_a)$ is constant on S_i because of (ii), and so $\text{lpp}(S_i) = \text{lpp}(I_a)$ for some $a \in S_i$ is well-defined. (Here lpp denotes the leading power products with respect to $\succ_{\overline{x}}$.)

The S_i's are called the segments of the Gröbner system. Depending on the context one can also assume that the segments are arbitrary constructible subsets (as e.g. in Manubens and Montes (2009)), or locally closed subsets of the special form

$$\left\{a \in \overline{K}^m : f_1(a) = 0, \dots, f_s(a) = 0,\ g_1(a) \neq 0, \dots, g_t(a) \neq 0\right\} = \mathbb{V}(f_1, \dots, f_s) \setminus \mathbb{V}\left(\prod g_j\right)$$

with $f_1, \dots, f_s, g_1 \dots, g_t \in K[\overline{a}]$ as in Weispfenning (1992). In a more algorithmic context one usually replaces S_i with some polynomial data in the parameters that determines S_i. Some authors (e.g. Suzuki and Sato (2006)) also drop condition (iii). If one requires $B_i \subset I$ then the Gröbner system is called faithful. From a faithful Gröbner system one can obtain a comprehensive Gröbner bases B simply by $B = \cup B_i$. Our focus is on Gröbner systems rather than on comprehensive Gröbner bases because we think that the decomposition of the parameter space is very important in the applications.

After Weispfenning (1992), the effort has gone in two directions. Weispfenning (2003) and other authors (Manubens and Montes, 2009) worked in the direction of obtaining a canonical discussion only associated to the given ideal and monomial order, focusing on nice properties of the discussion. Other authors (Kapur, 1995; Kalkbrenner, 1997; Suzuki and Sato, 2006, 2007; Nabeshima, 2006) fixed their objective on effectiveness and speed.

A common problem with algorithms for the computation of Gröbner systems is that, mainly due to the large number of segments generated, the interpretation of the output can become quite tedious for the user.

Therefore the main focus of this article is not on the efficiency of the algorithm but on computing a Gröbner system that has as few segments as possible and satisfies some additional nice properties, so that the compact output allows an easy interpretation and the algorithm is easy to use in applications. Thus for us the crucial topic is how to actually represent all the reduced Gröbner bases for varying $a \in \overline{K}^m$ in the most simple and canonical way on the computer.

There is a certain difficulty with (reduced) Gröbner systems: Let $S \subset \overline{K}^m$ be a locally closed subset such that $a \mapsto \text{lpp}(I_a)$ is constant on S and t an element of the minimal generating set of $\text{lpp}(S)$. For $a \in S$ let $g(a)$ denote the element of the reduced Gröbner basis of I_a with $\text{lpp}(g(a)) = t$. It is in general not possible to describe the function g on S by a single polynomial $p \in K[\overline{a}, \overline{x}]$. One reason for this can be that p might specialize to zero at a certain point $a \in S$, in other words, if we normalize p and consider it as element in $K(\overline{a})[\overline{x}]$ then $p(a, \overline{x})$ might not be defined for all $a \in S$ because some denominator specializes to zero. To avoid this kind of "singularities" we propose to use regular functions as in Wibmer (2007). We illustrate the above described phenomena with an example.

Example 1. Let $I = \langle ax + by, cx + dy \rangle \subset \mathbb{C}[a, b, c, d][x, y]$. We use a term-order with $x > y$. It is easy to see how the parameter space is partitioned according to lpp: