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Good bases for tame polynomials

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Abstract

An algorithm to compute a good basis of the Brieskorn lattice of a cohomologically tame polynomial is described. This algorithm is based on the results of C. Sabbah and generalizes the algorithm by A. Douai for convenient Newton non-degenerate polynomials.

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Introduction

Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ with $n \geq 1$ be a cohomologically tame polynomial function (Sabbah, 1998b). This means that no modification of the topology of the fibres of f comes from infinity. In particular, the set of critical points $\mathrm{C}(f)$ of f is finite. Then the reduced cohomology of the fibre $f^{-1}(t)$ for $t \notin \mathrm{C}(f)$ is concentrated in dimension n and equals $\mathbb{C}^\mu$ where μ is the Milnor number of f. Moreover, the n-th cohomology of the fibres of f forms a local system H^n on $\mathbb{C} \setminus \mathrm{D}(f)$ where $\mathrm{D}(f) = f(\mathrm{C}(f))$ is the discriminant of f. Hence, there is a monodromy action of the fundamental group $\Pi_1(\mathbb{C} \setminus \mathrm{D}(f), t)$ on H^n_t.

The Gauss–Manin system M of f is a regular holonomic module over the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$ with associated local system H^n on $\mathbb{C} \setminus \mathrm{D}(f)$. The Fourier transform $G := \widehat{M}$ of

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M is the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module defined by $\tau := \partial_t$ and $\partial_\tau = -t$. The monodromy T_∞ of M around $\mathrm{D}(f)$ can be identified with the inverse of the monodromy $\widehat{T_0}$ of G at 0. It turns out that ∂_t is invertible on M and hence G is a $\mathbb{C}[\tau,\theta]$-module where $\theta := \tau^{-1}$. A finite $\mathbb{C}[\tau]$- resp. $\mathbb{C}[\theta]$-submodule $L \subset G$ such that $L[\theta] = G$ resp. $L[\tau] = G$ is called a $\mathbb{C}[\tau]$- resp. $\mathbb{C}[\theta]$-lattice. The regularity of M at ∞ implies that G is singular at most in $\{0,\infty\}$ and where $0 := \{\tau = 0\}$ is regular and $\infty := \{\theta = 0\}$ of type 1. In particular, the V-filtration $V_\bullet$ on G at 0 consists of $\mathbb{C}[\tau]$-lattices.

The Brieskorn lattice $G_0 \subset G$ is a t-invariant $\mathbb{C}[\theta]$-submodule of G such that $G = G_0[\tau]$. Sabbah (1998b) proved that G_0 is a free $\mathbb{C}[t]$- and $\mathbb{C}[\theta]$-module of rank μ. In particular, G is a free $\mathbb{C}[\tau,\theta]$-module of rank μ. By definition, the spectrum of a $\mathbb{C}[\theta]$-lattice $L \subset G$ is the spectrum of the induced V-filtration $V_\bullet(L/\theta L)$ and the spectrum of f is the spectrum of G_0.

Sabbah (1998b) showed that there is a natural mixed Hodge structure on the moderate nearby cycles of G with Hodge filtration induced by G_0. This leads to the existence of good bases of the Brieskorn lattice. For a basis $\underline{\phi} = \phi_1,\ldots,\phi_\mu$ of a t-invariant $\mathbb{C}[\theta]$-lattice,

$$t \circ \underline{\phi} = \underline{\phi} \circ (A^{\underline{\phi}} + \theta^2\partial_\theta)$$

where $A^{\underline{\phi}} \in \mathbb{C}[\theta]^{\mu\times\mu}$. A $\mathbb{C}[\theta]$-basis $\underline{\phi}$ of G_0 is called good if $A^{\underline{\phi}} = A_0^{\underline{\phi}} + \theta A_1^{\underline{\phi}}$ where $A_0^{\underline{\phi}}, A_1^{\underline{\phi}} \in \mathbb{C}^{\mu\times\mu}$,

$$A_1^{\underline{\phi}} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix}$$

and $\phi_i \in V_{\alpha_i} G_0$ for all $i \in [1,\mu]$. One can read off the monodromy $T_\infty = \widehat{T}_0^{-1}$ from $A^{\underline{\phi}}$ immediately. The diagonal $\underline{\alpha} = \alpha_1,\ldots,\alpha_\mu$ is the spectrum of f and determines with $\mathrm{gr}_1^V A_0$ the spectral pairs of f. The latter correspond to the Hodge numbers of the above mixed Hodge structure.

Analogous results to those above were first obtained in a local situation where $f : (\mathbb{C}^n, \underline{0}) \longrightarrow (\mathbb{C}, 0)$ is a holomorphic function germ with an isolated critical point (Brieskorn, 1970; Sebastiani, 1970; Steenbrink, 1976; Pham, 1979; Varchenko, 1982; Saito, 1989). In this situation, the role of the Fourier transform is played by microlocalization and the algorithms in Schulze (2002, 2004a) compute A_0 and A_1 for a good $\mathbb{C}\{\{\theta\}\}$-basis of the (local) Brieskorn lattice. But Schulze (2004a) and Schulze (2002, 7.4–5) do not apply to the global situation.

Douai (1999) explained how to compute a good basis of G_0 if f is convenient and Newton non-degenerate using the equality of the V- and Newton filtration (Khovanskii and Varchenko, 1985; Sabbah, 1998b) and a division algorithm with respect to the Newton filtration (Douai, 1993; Briançon et al., 1989).

The intention of this article is to describe an explicit algorithm to compute a good basis of G_0 for an arbitrary cohomologically tame polynomial f. This algorithm is based on the following idea: