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Journal of Symbolic Computation 40 (2005) 1023–1038
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Generic initial ideals of points and curves

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Received 28 February 2004; accepted 25 January 2005
Available online 31 May 2005

Abstract

Let I be the defining ideal of a smooth complete intersection space curve C with defining equations of degrees a and b. We use the partial elimination ideals introduced by Mark Green to show that the lexicographic generic initial ideal of I has Castelnuovo–Mumford regularity $1 + ab(a-1)(b-1)/2$ with the exception of the case $a = b = 2$, where the regularity is 4. Note that $ab(a-1)(b-1)/2$ is exactly the number of singular points of a general projection of C to the plane. Additionally, we show that for any term ordering τ, the generic initial ideal of a generic set of points in $\mathbb{P}^r$ is a τ-segment ideal.

Keywords: Generic initial ideal; Curves; Points; Lexicographic term order

1. Introduction

Let $S = k[x_0, \ldots, x_r]$ where k is an algebraically closed field of characteristic zero and let τ be a term ordering on S. Let $I \subset S$ be a homogeneous ideal. There is a monomial ideal canonically associated with I, its generic initial ideal with respect to τ, denoted by $\text{gin}_\tau(I)$, or simply $\text{gin}_\tau I$. In this paper we study lexicographic generic initial ideals of curves and points via Green's partial elimination ideals.

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0747-7171/$ - see front matter
doi:10.1016/j.jsc.2005.01.009

For a smooth complete intersection curve C in $\mathbb{P}^3$, we show that the complexity of its lexicographic generic initial ideal, as measured by Castelnuovo–Mumford regularity, is governed by the geometry of a generic projection of C to $\mathbb{P}^2$.

Theorem 1.1. *Let C be a smooth complete intersection of hypersurfaces of degrees $a, b > 1$ in $\mathbb{P}^3$. The regularity of the lexicographic generic initial ideal of C is equal to*

$$\begin{cases} 1 + \frac{a(a-1)b(b-1)}{2} & \text{if } (a,b) \neq (2,2) \\ 4 & \text{if } (a,b) = (2,2). \end{cases}$$

Note that, apart for the exceptional case $a = b = 2$, the regularity of the lexicographic generic initial ideal is 1+ the number of nodes of the generic projection of C to $\mathbb{P}^2$. The statement of Theorem 1.1 generalizes Example 6.10 in Green (1998) which treats the special case where $a = b = 3$.

Macaulay's characterization of Hilbert functions, see for instance Theorem 4.2.10 in Bruns and Herzog (1993), implies that any ideal J is generated in degrees bounded by the largest degree of a generator of the corresponding lex-segment $\text{Lex}(J)$. Much more is true—Bigatti (1993), Hulett (1993) and Pardue (1996) showed the Betti numbers of J are bounded by those of $\text{Lex}(J)$. Let I be the ideal of C in Theorem 1.1. For such an ideal I one can compute the largest degree of a generator of $\text{Lex}(I)$. This has been done, for instance, by Bayer in his Ph.D. thesis (Proposition in II.10.4, Bayer, 1982) and by Chardin and Moreno-Socías (2002), and it turns out to be $\frac{a(a-1)b(b-1)}{2} + ab$. So the lexicographic generic initial ideal in Theorem 1.1 is not equal to the lex-segment ideal but nearly achieves the worst-case regularity for its Hilbert function. Moreover, as shown in Bermejo and Lejeune-Jalabert (1999), the extremal bound can only be achieved if C lies in a plane.

We also study the generic initial ideals of finite sets of points. Surprisingly, when X is a set of generic points its generic initial ideal is an *initial segment*.

Theorem 1.2. *Let I be the ideal of s generic points of $\mathbb{P}^n$. Then $\text{gin}_\tau I$ is equal to the τ-segment ideal $\text{Seg}_\tau(I)$ for all term orders τ. In particular, $\text{gin}_{\text{lex}} I$ is a lex-segment ideal.*

The genericity required in Theorem 1.2 is quite explicit: the conclusion holds for a set X of s points if there is a system of coordinates such that the defining ideal of X does not contain non-zero forms supported on $\leq s$ monomials. A special case of the result when $\tau = \text{revlex}$ is proved in Marinari and Ramella (1999).

For an introduction to generic initial ideals see Section 15.9 in Eisenbud (1995). Here we just recall:

Theorem 1.3 (*Galligo, Bayer–Stillman*). *Given a homogeneous ideal I and a term ordering τ on the monomials of S, there exists a dense open subset $U \subseteq \text{GL}_{r+1}(k)$ such that $\text{gin}_\tau I := \text{in}_\tau(g \cdot I)$ is constant over all $g \in U$ and $\text{gin}_\tau I$ is Borel-fixed.*

Recall also that, in characteristic 0, an ideal J is Borel-fixed if it is monomial and satisfies:

$$\text{if } m \text{ is a monomial, } x_i m \in J \implies x_j m \in J, \ \forall j \leq i.$$