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# Ideals with larger projective dimension and regularity

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ABSTRACT

We define a family of homogeneous ideals with large projective dimension and regularity relative to the number of generators and their common degree. This family subsumes and improves upon constructions given by Caviglia (2004) and McCullough (2011). In particular, we describe a family of three-generated homogeneous ideals, in arbitrary characteristic, whose projective dimension grows asymptotically as a power of the degree of the generators.



## 1. Introduction

Throughout this paper, let $K$ be a field of any characteristic and set $R = K[x_1, \ldots, x_n]$. We consider the following question of Stillman:

**Question 1.1** (Stillman, Peeva and Stillman (2009, Problem 3.14))**.** *Fix a sequence of natural numbers $d_1, \ldots, d_N$. Does there exist a number $p$ (independent of $n$) such that*

$$\mathrm{pd}(R/I) \leq p$$

*for all graded ideals $I$ with a minimal system of homogeneous generators of degrees $d_1, \ldots, d_N$?*

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This question is open in all but low degree cases. In Zhang (2011), Zhang's work on local cohomology modules in characteristic $p$ suggested that $\sum_{i=1}^{N} d_i$ was an upper bound for $\mathrm{pd}(R/I)$. In McCullough (2011), the second author showed this was false by producing a family of ideals whose projective dimensions far exceeded this bound. However, in the three-generated ideal case, these ideals had projective dimension of only $d+2$ where $d$ is the common degree of the generators. To the best of our knowledge there were no known ideals with three degree $d$ generators with larger projective dimension. Clearly then $d+2$ is a lower bound for any answer to the three-generated case of Stillman's Conjecture. We note that by the work of Burch (1968) and later Bruns (1976), it is natural to ask whether any three-generated ideals in degree $d$ have larger projective dimension than this.

In this paper we generalize the family of ideals given in McCullough (2011) to a larger family with much larger projective dimension. In the three-generated case, we produce a family of ideals with generators of degree $d$ and projective dimension larger than $\sqrt{d}^{\sqrt{d}-1}$. We therefore give a new lower bound for any answer to Stillman's question.

The paper is organized as follows. In Section 2 we recall some previous results and definitions. In Section 3 we define our family of ideals and compute its projective dimension. In Section 4 we compute some specific examples and show that this family subsumes two interesting families of ideals previously defined. We conclude with some computations and questions regarding the Castelnuovo–Mumford regularity of these ideals.

## 2. Preliminaries

Let $R = K[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_N)$ be a homogeneous ideal and set $d_i = \deg(f_i)$. Let $F_\bullet$ be the minimal graded free resolution of $R/I$. Then we may write

$$F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}},$$

where $R(-j)$ denotes a rank one free module with generator in degree $j$. In this case $F_0 = R$ and $F_1 = \bigoplus_{j=1}^{N} R(-d_j)$. The exponents $\beta_{i,j}$ are called the Betti numbers of $R/I$. We can define the projective dimension of $R/I$ as

$$\mathrm{pd}(R/I) = \max\{i \mid \beta_{i,j} \neq 0 \text{ for some } j\}.$$

Thus, Stillman's question can be rephrased by asking if $\mathrm{pd}(R/I)$ is bounded by a formula dependent only on $\beta_{1,j}$.

The Castelnuovo–Mumford regularity of $R/I$ is defined as

$$\mathrm{reg}(R/I) = \max\{j - i \mid \beta_{i,j} \neq 0 \text{ for some } i\}.$$

The Betti numbers are often displayed in matrix form called a Betti table. In the $(i, j)$ entry we put $\beta_{i,j-i}$. Thus, we can view the projective dimension of $R/I$ as the index of the last nonzero column in the Betti table and the regularity of $R/I$ as the index of the last nonzero row.

Let $\mathfrak{m}$ be the graded maximal ideal of $R$. We also denote the length of the maximal regular sequence on $R/I$ contained in $\mathfrak{m}$ by $\mathrm{depth}(R/I)$. Finally, we let $\mathrm{socle}(R/I)$ denote $\{x \in R/I \mid x\mathfrak{m} = 0\}$. To compute projective dimension, we make use of the graded version of the Auslander–Buchsbaum Theorem (see, e.g., Eisenbud (1995, Theorem 19.9)), so in order to show that $R/I$ has maximal projective dimension, we need only show that $\mathrm{socle}(R/I) \neq 0$.

Further motivating Question 1.1 is Problem 3.15 of Peeva and Stillman (2009) is an analog of Stillman's question for regularity: Is there a bound for $\mathrm{reg}(R/I)$ dependent only on $d_1, \ldots, d_N$? Caviglia showed that this question is equivalent to Question 1.1. See Engheta (2005), pages 11–14 for a nice explanation of this argument.

It is clear that there is an affirmative answer to Stillman's question when $N \leq 2$ or when $d_1 = \cdots = d_N = 1$. Eisenbud and Huneke (in unpublished work) verified the case $N = 3$ and $d_1 = d_2 = d_3 = 2$ by showing that for ideals $I$ generated by three quadrics, $\mathrm{pd}(R/I) \leq 4$. In Engheta (2010), Engheta verified the case $N = 3$ and $d_1 = d_2 = d_3 = 3$ giving $\mathrm{pd}(R/I) \leq 36$ for this case. This bound is likely not tight as the largest known projective dimension of $R/I$ for an ideal $I$ generated by