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Computing the multiplicity structure of an isolated singular solution: Case of breadth one

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ABSTRACT

We present an explicit algorithm to compute a closed basis of the local dual space of $I = (f_1, \ldots, f_t)$ at a given isolated singular solution $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ when the Jacobian matrix $J(\hat{\mathbf{x}})$ has corank one. The algorithm is efficient both in time and memory use. Moreover, it can be modified to compute an approximate basis if the coefficients of $f_1, \ldots, f_t$ and $\hat{\mathbf{x}}$ are only known with limited accuracy.

1. Introduction

Motivation and problem statement. Consider an ideal I generated by a polynomial system $F = \{f_1, \ldots, f_t\}$, where $f_i \in \mathbb{C}[x_1, \ldots, x_s]$, $i = 1, \ldots, t$. For a given isolated singular solution $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ of F, suppose Q is the isolated primary component whose associate prime is $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$. In (Wu and Zhi, 2008), we used the symbolic–numeric method based on the geometric jet theory of partial differential equations introduced in (Reid et al., 2003; Zhi and Reid, 2004; Bonasia et al., 2004) to compute the index ρ, the minimal nonnegative integer such that $P^\rho \subseteq Q$, and the multiplicity $\mu = \dim(\mathbb{C}[\mathbf{x}]/Q)$, where $Q = (I, P^\rho)$. A basis for the local dual space of I at $\hat{\mathbf{x}}$ is obtained from the null space of the truncated coefficient matrix of the involutive system. The size of these coefficient matrices is bounded by $t\binom{\rho+s}{s} \times \binom{\rho+s}{s}$ which will be very big when ρ or s is large. In general, $\rho \leq \mu$. However, when the corank of the Jacobian matrix is one, then $\rho = \mu$, which is also called the breadth one case in (Dayton and Zeng, 2005; Dayton et al., 2009). The size of the matrices grows extremely fast with the multiplicity μ. As pointed out in (Zeng, 2009), the matrix size becomes the main bottleneck that slows down the overall computation. This is the main motivation for us to consider whether we can compute the multiplicity structure of $\hat{\mathbf{x}}$ efficiently in this worst case.

In (Dayton and Zeng, 2005; Dayton et al., 2009), they presented an efficient algorithm for computing a dual basis for the breadth one case by solving a deflated system of size roughly $(\mu t) \times$

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(μs). A general construction of a Gauss basis of differential conditions at a multiple point was also given in (Marinari et al., 1996, Section 4.3), the breadth one case is just a special case. The size of linear systems they constructed is bounded by $(\mu t) \times (\mu s)$, and they assumed that I is a zero dimensional system. In (Stetter, 2004, Section 8.5), an algorithmic approach for determining a basis of the local dual space incrementally was stated and some examples were given to show that only a sizable number of free parameters are needed when we compute the k-th order differential condition.

Main contribution. In the breadth one case, following Stetter's arguments and smart strategies given in (Stetter, 2004, Section 8.5), we prove that the number of free parameters used in computing each order of the differential condition of I at $\hat{\mathbf{x}}$ can be reduced to $s-1$. So that we can compute the multiplicity structure of an isolated multiple zero $\hat{\mathbf{x}}$ very efficiently by solving $\mu-2$ linear systems with size bounded by $t \times (s-1)$. Moreover, during the computation, we only need to store polynomials, the LU decomposition of the last $s-1$ columns of the Jacobian matrix and the computed differential operators. Therefore, in the breadth one case, both storage space and execution time for computing a closed basis of the local dual space are reduced significantly. Furthermore, we modify the algorithm for computing an approximate basis when singular solutions and polynomials are only known approximately.

Structure of the paper. Section 2 is devoted to recalling some notations and well-known facts. In Section 3, we prove that for the breadth one case, a closed basis of the local dual space of I at $\hat{\mathbf{x}}$ can be constructed incrementally by checking whether a differential operator parameterized by $s-1$ variables is consistent with polynomials in I. In Section 4, we describe an algorithm for computing a closed basis of the local dual space of I at $\hat{\mathbf{x}}$ and the multiplicity μ. If I and $\hat{\mathbf{x}}$ are only known with limited accuracy, then we modify the symbolic algorithm by introducing one more parameter and using singular value decomposition or LU decomposition with pivoting to ensure the numeric stability of the algorithm. Three examples are given to demonstrate that our algorithms are applicable to positive dimensional systems, analytic systems and polynomial systems with irrational or approximate coefficients. The complexity analysis and experiments are done in Section 5. We mention some ongoing research in Section 6.

2. Preliminaries

Suppose we are given an isolated multiple root $\hat{\mathbf{x}}$ of the polynomial system $F = \{f_1, \ldots, f_t\}$ with multiplicity μ and index ρ.

Let $D(\alpha) = D(\alpha_1, \ldots, \alpha_s) : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ denote the differential operator defined by:

$$D(\alpha_1, \ldots, \alpha_s) := \frac{1}{\alpha_1! \cdots \alpha_s!} \partial x_1^{\alpha_1} \cdots \partial x_s^{\alpha_s},$$

for non-negative integer array $\alpha = [\alpha_1, \ldots, \alpha_s]$. We write $\mathfrak{D} = \{D(\alpha),\ |\alpha| \geq 0\}$ and denote by $\mathrm{Span}_{\mathbb{C}}(\mathfrak{D})$ the $\mathbb{C}$-vector space generated by $\mathfrak{D}$ and introduce a morphism on $\mathfrak{D}$ that acts as "integral":

$$\Phi_j(D(\alpha)) := \begin{cases} D(\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_s), & \text{if } \alpha_j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

As a counterpart of the anti-differentiation operator Φ_j, we define the differentiation operator Ψ_j as

$$\Psi_j(D(\alpha)) := D(\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_s).$$

Definition 1. Given a zero $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ of an ideal $I = (f_1, \ldots, f_t)$, we define the local dual space of I at $\hat{\mathbf{x}}$ as

$$\triangle_{\hat{\mathbf{x}}}(I) := \{L \in \mathrm{Span}_{\mathbb{C}}(\mathfrak{D}) \,|\, L(f)|_{\mathbf{x}=\hat{\mathbf{x}}} = 0,\ \forall f \in I\}. \tag{1}$$

The vector space $\triangle_{\hat{\mathbf{x}}}(I)$ and conditions equivalent to $L(f)|_{\mathbf{x}=\hat{\mathbf{x}}} = 0,\ \forall L \in \triangle_{\hat{\mathbf{x}}}(I)$ are also called *Max Noether space* and *Max Noether conditions* in Möller and Tenberg (2001) respectively.

For a non-negative integer k, $\triangle_{\hat{\mathbf{x}}}^{(k)}(I)$ consists of differential operators in $\triangle_{\hat{\mathbf{x}}}(I)$ with the differential order bounded by k. We have that $\dim_{\mathbb{C}}(\triangle_{\hat{\mathbf{x}}}(I)) = \mu$, where μ is the multiplicity of the zero $\hat{\mathbf{x}}$.