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The existence, uniqueness and global exponential stability of periodic solution for a coupled system on networks with time delays

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ABSTRACT

This paper investigates a type of coupled system on networks with time delays (CSND). By the combined method of graph theory, coincidence degree theory and Lyapunov function method, we establish sufficient conditions for the existence, uniqueness and global exponential stability of periodic solution of CSND. Moreover, the conditions we obtain are easy to be verified. Finally, we give a numerical example to illustrate the effectiveness of the results developed.

1. Introduction

In the real world, periodic movements arise in different areas, hence the problems of periodic solutions have been widely studied. Among them, the existence and stability of periodic solutions for various ordinary or functional differential equations have been extensively studied due to their importance, for example, see [1–14]. Many theorems and methods have been applied to research the existence of periodic solutions, such as various fixed point theorems, Lyapunov function method, the continuation theorem of coincidence degree, the upper and lower solutions method. In [2,3,9–14], on the basis of coincidence degree theory, the existence of periodic solutions is obtained. Furthermore, in [2,3], the authors also obtained global attractivity of periodic solution by using Lyapunov functional method and coincidence degree theory.

On the other hand, coupled systems on networks have attracted many scholars' attention and some results on the dynamical behaviors have been reported in the literature, for example, see [15–34] and references therein. It is worth pointing out that by using graph theory and Lyapunov function method, Li and Shuai [17,31] developed a systematic method to obtain the global asymptotical stability for coupled systems of ordinary differential equations on networks. By using this method, many sufficient conditions for global stability of different kinds of coupled systems on networks have been given, and we here mention only [20–27]. But most work is about the stability of equilibrium point. However, since systems can be affected by some periodic environmental factors in the real world, it is reasonable to study coupled systems on networks with periodic coefficients. In [28–30], by applying some analysis techniques and Lyapunov functional method, the authors studied the existence and global exponential stability of periodic solution for neural networks. But the dynamics of coupled systems on networks is hard to study due to the complexity of these systems. An effective way to obtain the global dynamics is to model coupled systems in digraphs. As far as we know, there are few results on the existence and global exponential stability of the periodic solutions of coupled systems on networks by using graph theory.

Considering the facts above, in this paper, we focus on the existence and global exponential stability of periodic solution for a type of coupled system on networks with time delays (CSND). The model we study can be constructed in digraph as follows. Given a digraph $\mathcal{G}$ with $l\,(l \ge 2)$ vertices, assume that the dynamics of each vertex system is described by the following differential equations with delays:

$$\dot{x}_k(t) = A(t)x_k(t) + f_k(t, x_k(t),\ x_k(t-\tau_k)), \quad t \ge 0, \quad 1 \le k \le l,$$

where $x_k(t) = (x_{k1}(t), x_{k2}(t), \ldots, x_{km}(t))^T$, $f_k(t, x, y) \in C(\mathbb{R}^+ \times \mathbb{R}^m \times$

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$\mathbb{R}^m, \mathbb{R}^m)$, $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{m\times m})$, and $f_k(t, x, y) = f_k(t+T, x, y)$, $A(t+T) = A(t)$ for some $T > 0$, τ_k is the time delay in the k-th subsystem. Suppose that the strength of the h-th subsystem on the k-th subsystem is described by $g_{kh}(x_h(t-\tau_h))$ then we get the following CSND:

$$\dot{x}_k(t) = A(t)x_k(t) + f_k(t, x_k(t), x_k(t-\tau_k)) + \sum_{h=1}^{l} g_{kh}(x_h(t-\tau_h)), \quad t \ge 0, \quad 1 \le k \le l. \tag{1}$$

Throughout this paper, we always suppose that $g_{kh}(0) = 0$.

Assume that system (1) is supplemented with the initial value

$$x(t) = \phi(t), \quad -\tau \le t \le 0, \tag{2}$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_l(t))^T$, $\phi_k(t) \in C([-\tau, 0], \mathbb{R}^m)$, and $\tau = \max\{\tau_1, \tau_2, \dots, \tau_l\}$.

The contributions and novelties of the current work are as follows:

1. Different from the ways in papers [2,3,9–14,29], the method of estimating the bound of periodic solutions to the auxiliary equation of system (1) is Lyapunov function method and graph theory, which avoids analyzing the bound straight.
2. By employing graph theory and constructing suitable Lyapunov function, we obtain sufficient conditions for the global exponential stability of periodic solution to system (1).
3. The conditions we obtain are easy to be checked.

The paper is organized as follows: In Section 2, sufficient conditions of the existence of periodic solutions are obtained. Then in Section 3, we derive that the periodic solution is unique and globally exponentially stable. Finally, a numerical example is given to show the correctness of the theory.

2. The existence of periodic solutions to system (1)

In this section, we mainly discuss the existence of periodic solutions of system (1). In our proof we will use the following basic lemmas on coincidence degree and graph theory. The concepts concerning the coincidence degree and graph theory are given in Appendix.

Lemma 1 (*Li and Shuai [17]*). *Assume $l \ge 2$. Let c_k denote the cofactor of the k-th diagonal element of Laplacian matrix of $(\mathcal{G}, A)$. Then the following identity holds:*

$$\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(y_k, y_h) = \sum_{Q\in\mathbb{Q}} W(Q) \sum_{(s,r)\in E(\mathfrak{C}_Q)} F_{rs}(y_r, y_s).$$

Here $F_{rs}(y_r, y_s)$, $1 \le r, s \le l$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(Q)$ is the weight of Q, and $\mathfrak{C}_Q$ denotes the directed cycle of Q. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for $k = 1, 2, \dots, l$.

Lemma 2 (*West [35]*). *Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose that the following conditions hold.*

(Y1) *For each $\lambda \in (0, 1)$, $\forall\, x \in \partial\Omega \cap DomL$, $Lx \ne \lambda Nx$;*
(Y2) *For each $x \in \partial\Omega \cap KerL$, $QNx \ne 0$;*
(Y3) *$deg_B\{JQN, \Omega \cap KerL, 0\} \ne 0$, where B denotes the Brouwer degree.*

Then the equation $Lx=Nx$ has at least one solution lying in Dom $L \cap \bar{\Omega}$.

Now we are in the position to prove the existence of periodic solutions of system (1). For the sake of simplicity, we use the following notations in the sequel. $\mathbf{L} = \{1, 2, \dots, l\}$. $|\cdot|$ denotes the Euclidean norm for vectors.

Theorem 1. *Suppose that there exist constants $a \ge 0$, $b \ge 0$, $d > 0$, $\rho > 0$, and a matrix $A = (a_{kh})_{l\times l}$, $a_{kh} \ge 0$, such that the following conditions hold:*

A1. $|f_k(t, x, y)| \le a|x| + b|y| + d$, $\forall\ (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^m$, $k \in \mathbb{L}$.
A2. $\lambda_{\max}(\frac{A(t)+A^T(t)}{2}) \le -\rho$, $\forall\ t \in \mathbb{R}^+$, *where* $\lambda_{\max}(B)$ *is the maximum eigenvalue of matrix B,* A^T *is the transposed matrix of A.*
A3. $|g_{kh}(x)| \le a_{kh}|x|$, $\forall\ x \in \mathbb{R}^m$.
A4. $(\mathcal{G}, A)$ *is strongly connected.*

Then system (1) has at least one T-periodic solution if

$$\rho > \max\{a + b + \max_{k\in\mathbb{L}} \sum_{h=1}^{l} a_{kh},\ a + b + \frac{1}{2}\max_{k\in\mathbb{L}} \sum_{h=1}^{l} a_{kh} + \frac{1}{2}\max_{h\in\mathbb{L}} \sum_{k=1}^{l} a_{kh}\}.$$

Proof. We divide the proof into three steps.

Step 1: Let $X = Z = \{x = (x_1, x_2, \dots, x_l)^T \in C(\mathbb{R}^+, \mathbb{R}^{ml}): x(t+T) = x(t)\}$ with the norm

$$\| x \| = \left(\sum_{k=1}^{l} \max_{t\in[0,T]} \left(\sum_{i=1}^{m} |x_{ki}(t)|^2 \right) \right)^{1/2}.$$

Then X and Z are Banach spaces. Define operators L and N as follows:

$$L: \mathrm{Dom}L \subset X \to X, \quad L(x_1, x_2, \dots, x_l)^T = (x_1', x_2', \dots, x_l')^T$$

and

$$N: X \to X,\ N\begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix} = \begin{pmatrix} A(t)x_1(t) + f_1(t, x_1(t), x_1(t-\tau_1)) + \sum_{h=1}^{l} g_{1h}(x_h(t-\tau_h)) \\ \vdots \\ A(t)x_l(t) + f_l(t, x_l(t), x_l(t-\tau_l)) + \sum_{h=1}^{l} g_{lh}(x_h(t-\tau_h)) \end{pmatrix},$$

then $\mathrm{Ker}\, L = \{x \in X: x = c \in \mathbb{R}^{ml}\}$, $\mathrm{Im}\, L = \{z \in Z: \int_0^T z(t)\mathrm{d}t = 0\}$ is closed in Z, and

$$\mathrm{Dim\ Ker}\ L = ml = \mathrm{Codim\ Im}\ L.$$

Therefore the operator L is a Fredholm mapping of index 0.

Let the project operators P and Q as follows, respectively:

$$Px = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t, \quad x \in X, \qquad Qz = \frac{1}{T}\int_0^T z(t)\,\mathrm{d}t, \quad z \in Z.$$

Hence,

$$\mathrm{Im}\ P = \mathrm{Ker}\ L, \quad \mathrm{Im}\ L = \mathrm{Ker}\ Q = \mathrm{Im}(I - Q).$$

Furthermore, the generalized inverse (of L) $K_p: \mathrm{Im}\, L \to \mathrm{Ker}\, P \cap \mathrm{Dom}L$ is

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore,