Omega 48 (2014) 1–9

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

Stochastic efficiency analysis with a reliability consideration

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ARTICLE INFO

Article history:
Received 3 May 2012
Accepted 3 April 2014
Processed by B. Lev
Available online 14 April 2014

Keywords:
Data Envelopment Analysis
Performance measurement
Stochastic efficiency analysis
Chance constraint
Value-at-risk

ABSTRACT

Stochastic Data Envelopment Analysis (DEA) models have been introduced in the literature to assess the performance of operating entities with random input and output data. A stochastic DEA model with a reliability constraint is proposed in this study that maximizes the lower bound of an entity's efficiency score with some pre-selected probability. We define the concept of stochastic efficiency and develop a solution procedure. The economic interpretations of the stochastic efficiency index are presented when the inputs and outputs of each entity follow a multivariate joint normal distribution.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric method used to evaluate the performance of a set of operating entities called decision making units (DMUs) that consume similar inputs and create similar outputs. It has been widely applied in areas such as healthcare, agriculture and banking as well as assessing low carbon supply chains. Cooper et al. [10] provided an introduction of the various DEA models. Cook et al. [6] discussed the selection of a DEA model. The reader is referred to Cook and Seiford [5] and Liu et al. [17,18] for extensive reviews of DEA's development and applications.

Traditionally, the efficiency score of a DMU is defined as the ratio of the multiplier-weighted sum of its outputs to the multiplier-weighted sum of its inputs. The constant returns-to-scale DEA model, namely, the CCR DEA model [4], computes the efficiency index of a DMU, which is the maximum efficiency score in terms of the input and output multipliers. Any DMU with an efficiency index of one is rated as *CCR efficient* in the sense that it is not dominated by any observations or their linear combinations. The efficiency index of an inefficient DMU is less than one and reveals the proportional decrease necessary in its inputs to reach the estimated efficiency frontier, which is spanned by the efficient units.

It is widely acknowledged that variability and uncertainty are associated with the input and output data of a production process due to its inherent stochastic nature or specification errors [1]. Land et al. [14] gave convincing examples in agriculture, manufacturing, product development, education, health care and military for which it is necessary to incorporate stochastic variation of data in the concept of "efficiency". As a consequence, both multiplier and envelopment DEA models have been generalized to deal with stochastic inputs and outputs. The concepts of *dominance* and *efficiency* are extended to the stochastic domain in these models, where chance-constrained programming is applied to model the production frontier defined with stochastic inputs and outputs.

Land et al. [14] proposed a stochastic efficiency analysis formulation in envelopment form where a chance constraint is placed on every output category. In this study we focus on stochastic DEA models in multiplier form as they explicitly take into account the correlations among input and output data within every DMU, which are generally considered more important than dependencies among the observed DMUs but are ignored in envelopment models.

Cooper et al. [8,9], Huang and Li [12,13] and Li [16] developed *joint* stochastic efficiency analysis models where probabilistic efficiency dominance is established via a joint chance constraint. No computational results have been reported in the literature possibly due to the strong intractability of these models.

We next examine two multiplier form stochastic DEA models with a marginal chance constraint on every DMU. The following "satisficing" DEA model was presented in Cooper et al. [7]:

$$\pi_o^* = \max_{\mathbf{u},\mathbf{v}} P\left\{\frac{\mathbf{u}^T\tilde{\mathbf{y}}_o}{\mathbf{v}^T\tilde{\mathbf{x}}_o} \ge 1\right\}$$
$$\text{s.t.}$$
$$P\left\{\frac{\mathbf{u}^T\tilde{\mathbf{y}}_j}{\mathbf{v}^T\tilde{\mathbf{x}}_j} \le 1\right\} \ge \alpha_j, \quad j \in N,$$
$$\mathbf{u} \ge \mathbf{0},\ \mathbf{v} \ge \mathbf{0}. \quad (1)$$

In the model, it is assumed that every unit in the set of DMUs, $N = \{1, 2, \ldots, n\}$, consumes resources in m categories and creates

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http://dx.doi.org/10.1016/j.omega.2014.04.001
0305-0483/

products or services in s categories. P means "Probability", $\tilde{\mathbf{y}}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{sj})^T$ and $\tilde{\mathbf{x}}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})^T$ represent, respectively, the vectors of stochastic output and input values of DMU $j \in N$, while $\mathbf{u} \in \mathbb{R}^s$ and $\mathbf{v} \in \mathbb{R}^m$ are non-negative virtual multipliers to be determined by solving the above model for DMU o, which is the DMU under evaluation. Throughout this paper, it is assumed that $\tilde{y}_{rj}$ and $\tilde{x}_{ij}$ are continuous random variables for any $r = 1, 2, \ldots, s$ and any $i = 1, 2, \ldots, m$. $\alpha_j \in (0, 1)$ is pre-selected and is the minimum probability required to fulfill the corresponding chance constraint.

We note that model (1) is adapted from the traditional CCR DEA model [4] and falls in the class that Charnes and Cooper [3] refer to as "P-models". As Charnes and Cooper suggested, the objective of a "P-model" can be linked to the concept of "satisficing" defined by Simon [21]. Along this perspective, the unity in the objective function of model (1) can be interpreted as an *aspiration level*, while model (1) maximizes the likelihood for the efficiency score of DMU o to achieve this aspiration level.

Assuming that the random outputs and inputs of each DMU j follow a multivariate normal distribution with a mean vector $(\overline{\mathbf{y}}_j^T, \overline{\mathbf{x}}_j^T)^T$ and a variance–covariance matrix $\mathbf{\Lambda}_j$, Olesen and Petersen [20] developed a model that optimizes the rate at which the mean input vector for the DMU under evaluation has to decrease in order to achieve efficiency. The original formulation presented by Olesen and Petersen [20] has a typo. The model after the necessary correction is presented as follows:

$$\begin{aligned}
&\theta_o^* = \max_{\mathbf{u},\mathbf{v}} \mathbf{u}^T\overline{\mathbf{y}}_o + \Phi^{-1}(\alpha_o)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\mathbf{\Lambda}_o(\mathbf{u}^T, -\mathbf{v}^T)^T}\\
&\quad \text{s.t.}\\
&\quad \mathbf{v}^T\overline{\mathbf{x}}_o = 1,\\
&\quad \mathbf{u}^T\overline{\mathbf{y}}_j - \mathbf{v}^T\overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\mathbf{\Lambda}_j(\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0, \quad j \in N,\\
&\quad \mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}.
\end{aligned} \tag{2}$$

In the model, $\Phi()$ is the standard normal distribution function and $\Phi^{-1}()$ its inverse.

As will be illuminated in the next section, the stochastic efficiency index π_o^* given by model (1) is not a radial measure. In contrast, model (2) returns a radial measure θ_o^* and reduces to the CCR DEA model when there is no variability in input and output data. Consequently, (2) is a general model with CCR DEA model as a special case. However, our subsequent analysis will show that model (2) does not necessarily return a correct stochastic efficiency index. In this study, we propose a stochastic efficiency analysis model that corrects this shortcoming of model (2) using the concept of aspiration level introduced in model (1). We next analyze an example to motivate the study.

2. A motivating example

Under the assumption of joint normality model (1) can be rewritten as follows:

$$\begin{aligned}
&\vartheta_o^* = \max_{\mathbf{u},\mathbf{v},\vartheta} \vartheta\\
&\quad \text{s.t.}\\
&\quad \mathbf{u}^T\overline{\mathbf{y}}_o - \mathbf{v}^T\overline{\mathbf{x}}_o - \vartheta\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\mathbf{\Lambda}_o(\mathbf{u}^T, -\mathbf{v}^T)^T} \ge 0,\\
&\quad \mathbf{u}^T\overline{\mathbf{y}}_j - \mathbf{v}^T\overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\mathbf{\Lambda}_j(\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0, \quad j \in N,\\
&\quad \mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}.
\end{aligned} \tag{3}$$

where $\pi_o^* = \Phi(\vartheta_o^*)$.

Models (2) and (3) are interpreted in this section using an example of three DMUs with a single output and a single input that follow a joint normal probability distribution. As shown in Olesen and Petersen [20], each chance constraint $\mathbf{u}^T\overline{\mathbf{y}}_j - \mathbf{v}^T\overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\mathbf{\Lambda}_j(\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0$ in these two models generates a supporting hyperplane to a confidence region of DMU j at some confidence level related to the chance constraint probability level α_j. Olesen and Petersen [20] further noted that the production possibility set (PPS) is spanned by these confidence regions in the input-output space. We present the motivating example in Figs. 1 and 2 without discussing the mathematical details. The confidence region of DMU j in both figures is an ellipsoid denoted by $D_j(\alpha_j)$ j=1, 2, 3, $\alpha_j > 50\%$, with the mean input and output $(\overline{x}_j, \overline{y}_j)$ of DMU j at the center, where the size of the region is derived from the probability level α_j used in the $j+1$ th chance constraint in model (3). The straight line in the two figures spanned by ellipsoid $D_1(\alpha_1)$ is the production frontier.

The other ellipsoids in the figures are adjusted confidence regions for DMU 2, the DMU under evaluation. These adjusted regions are denoted by $D_2'(q, \beta)$ with the mean output $\overline{y}_2$ and the contracted mean input $q\overline{x}_2$ from DMU 2 at the center, where $q \in (0, 1]$ is a radial contraction rate of the mean input vector $\overline{x}_2$

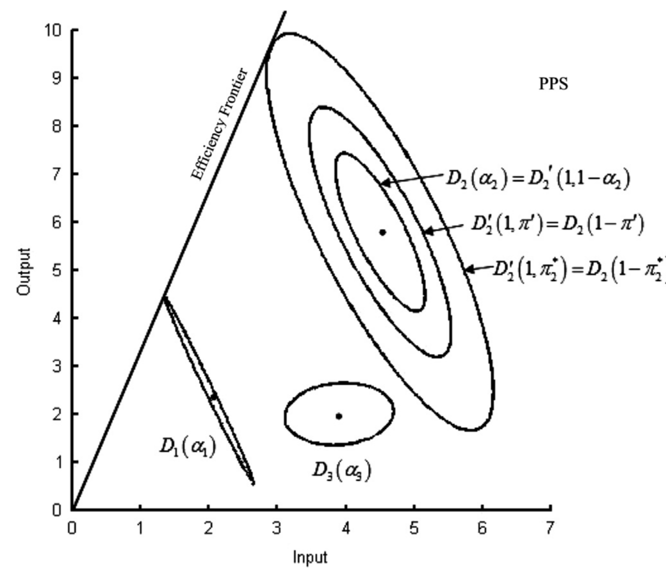

Fig. 1. Confidence regions used in models (2) and (3).

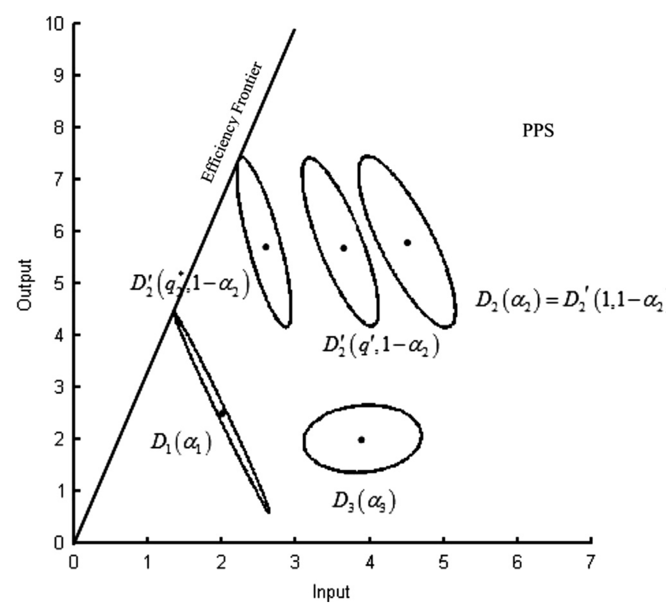

Fig. 2. Confidence regions used in the proposed model.