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On approximation of the fixed charge transportation problem

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ABSTRACT

In this paper we present a new approximation for computing lower bound for the fixed charge transportation problem (FCTP). The lower bounds thus generated delivered 87% optimal solutions for 56 randomly generated small, up to 6×10 in size, problems in an experimental design. For somewhat larger, 10×10 and 10×15 size problems, the lower bounds delivered an average error of 5%, approximately, using a fraction of CPU times as compared to CPLEX to solve these problems. The proposed lower bound may be used as a superior initial solution with any other existing branch-and-bound method or tabu search heuristic procedure to enhance convergence to the optimal solution for large size problems which cannot be solved by CPLEX due to time constraints.

1. Introduction

The fixed charge transportation problem (FCTP) is a variant of the transportation problem (TP), which arises when both variable and fixed costs are present. The FCTP is formulated as follows:

$$\mathbf{P}: \text{Minimize} \quad Z = \sum_{i=1}^{S} \sum_{j=1}^{D} (c_{ij}x_{ij} + f_{ij}y_{ij}) \tag{1}$$

$$\text{Subject to} \quad \sum_{j=1}^{D} x_{ij} = a_i, \qquad i = 1, 2, \ldots, S, \tag{2}$$

$$\sum_{i=1}^{S} x_{ij} = b_j, \qquad j = 1, 2, \ldots, D, \tag{3}$$

where

$$y_{ij} = \begin{cases} 1 & \text{if } x_{ij} > 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases} \quad x_{ij} \geq 0, \; i = 1, 2, \ldots, S; j = 1, \; 2, \ldots, D;$$

a_i represents the supply at supplier i ($i = 1, 2, \ldots, S$), b_j is the demand at customer j ($j = 1, 2, \ldots, D$), and x_{ij} is the number of units shipped by supplier i to customer j at shipping cost per unit c_{ij} plus fixed cost f_{ij}, assumed for opening this route.

Over the decades numerous proposals have been made to obtain either exact or approximate solutions for the FCTP (e.g., [1,2,4,21–23]). It is known that the optimal solution of FCTP occurs at an extreme point of the feasible region [15]. Based on this property, many researchers have made claims of computational success in generating the optimal solution. Among the many methods developed, only two guarantee an optimal solution: the stage-ranking method [24,27] and the branch-and-bound method [26,28]. Both of these methods are based on enumeration procedures and on a comparison of objective function values for a specified domain of distributions. The exact method of ranking extreme points requires analyzing a large domain of allocation distributions, while the effort required to solve an FCTP using exact branch-and-bound method grows exponentially with the size of the problem.

The above-mentioned methods are constrained by limits on computer time. Because of this limitation, some authors [5,6,10–12,18,32] have turned to efficient heuristic algorithms for solving FCTPs. Aguado [7] proposed an approach based on intensive use of the Lagrangean relaxation techniques. Sun et al. [29] provided a tabu search heuristic procedure. Glover and Kochenberger [13] presented a parametric approach for solving fixed-charge problems and they evaluated it by reference to transportation networks. Klose [19] presented algorithms for solving the single-sink FCTP whereas Jawahar and Balaji [16] solved the FCTP with a two stage supply chain distribution problem function. Other authors [14,17] have used the spanning tree-based genetic algorithm. Classical branch-and-bound methods have been applied to specific real-world applications by several authors [9,25,31]. Adlakha et al. [3] developed an analytical method that starts with a linear formulation of the problem and converges to an optimal solution by sequentially separating the fixed costs and finding a direction to improve the value of the linear formulation while continually tightening the

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lower and upper bounds. Schrenk et al. [30] analyzed degeneracy characterizations for the transportation paradox in linear transportation problems and the fixed charge problem.

The purpose of this paper is to provide an effective lower bound by adopting an approximation of the FCTP's objective function. The proposed lower bound is found to be much superior to the linear bound developed by Balinski [8] and yields optimal solutions for 87% randomly generated 3×5 to 6×10 problems in an experimental design. Experiments with somewhat larger, 10×10 and 10×15 in size, problems also support desirability of using the lower bounds for FCTP approximation when the CPU times are compared with those for finding exact solutions using an NLP solver. The rest of the paper is structured as follows. In Section 2 we present a lower bound formulation for the FCTP optimal value. Section 3 presents an example to illustrate the proposed approximation. In Section 4, we carry out a computational study and present percentage errors in the optimal solution value as obtained by the lower bounds and the corresponding FCTP values. In Section 5, we further discuss our proposal and, based on our computational experiments, make recommendations for solving an FCTP along with the limitations. Conclusions follow in Section 6 along with possible directions for future research.

2. Approximation to FCTP lower bound

In this Section, we develop an approximation of an FCTP in contrast to the linear approximation as provided by Balinski [8]. Balinski proposed relaxing the integer restriction on y_{ij}, with the property that $y_{ij}=x_{ij}/m_{ij}$ where $m_{ij}=\min(a_i, b_j)$. This relaxed transportation problem of an FCTP, problem **B**, is a classic TP with unit transportation costs $C_{ij}=c_{ij}+f_{ij}/m_{ij}$. Balinski shows that the optimal solution $\{x^B_{ij}\}$ to problem **B** provides a lower bound to the optimal value of FCTP, *i.e.*, $Z(\mathbf{B})=\sum\sum C_{ij}\,x^B_{ij}\le Z^*(\mathbf{P})\le\sum\sum(c_{ij}x^B_{ij}+f_{ij}y^B_{ij})=Z_B(\mathbf{P})$.

We propose an approximation, $\mathbf{P_L}$, for FCTP costs as proposed in Fig. 1 where, for the sake of convenience, x, f, c, and m represent x_{ij}, f_{ij}, c_{ij}, and m_{ij}, respectively. The idea here is to estimate FCTP costs more closely than the linear approximation provided by Balinski, in order to improve the solution estimate for the FCTP.

$$\text{Define}\quad P_L(x)=\alpha\sqrt{x}+\beta x+k, \tag{4}$$

where

$$P_L(0)=0, \tag{5}$$

$$P_L(m)=f+cm, \tag{6}$$

$$P'_L(m)=c. \tag{7}$$

Eqs. (5) and (6) ensure that the proposed curve, $P_L(x)$, starts at the origin and equals the FCTP cost at m. Eq. (7), where P'_L denotes the derivative of P_L, ensures that $P_L(x)$ is tangential to FCTP cost

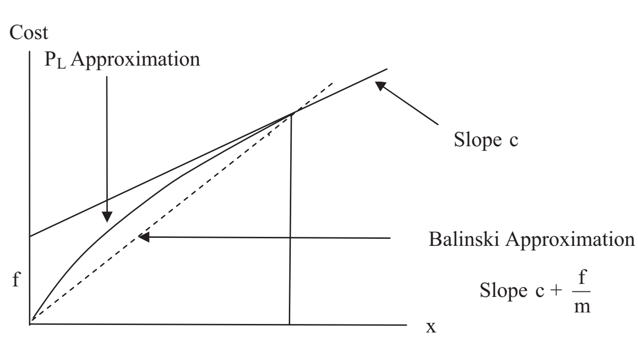

Fig. 1. Approximation for Lower Bounds to FCTP Costs.

function at $x=m$. Since $P_L(x)$ is concave in x, being tangential at $x=m$ helps us select the function in the $P_L(x)$ family such that it is closest to FCTP cost. We use these equations to determine coefficients in $P_L(x)$.

$P_L(0)=0\Rightarrow k=0.$

Eqs. (6) and (7) yield

$$\alpha\sqrt{m}+\beta m=f+cm$$

and

$$\frac{\alpha}{2\sqrt{m}}+\beta=c$$

Solving these two equations, we get

$$\alpha=\frac{2f}{\sqrt{m}}$$

and $\beta=c-(f/m)$

Therefore,

$$P_L(x)=\frac{2f}{\sqrt{m}}\sqrt{x}+\left\{c-\frac{f}{m}\right\}x.$$

Theorem 1. The approximation curve, $P_L(x)$, lies entirely above the Balinski linear approximation and below the FCTP cost line.

Proof: The proof is somewhat obvious because of the way the approximation curve $P_L(x)$ is defined. We nevertheless carry out the proof to verify the calculations of coefficients α and β.

$$\begin{aligned}&\text{Consider}\quad P_L(x)-B(x)\\&=\tfrac{2f}{\sqrt{m}}\sqrt{x}+\left\{c-\tfrac{f}{m}\right\}x-\left\{c+\tfrac{f}{m}\right\}x\\&=\tfrac{2f}{\sqrt{m}}\sqrt{x}-\tfrac{2f}{m}\,x\\&=\tfrac{2f}{\sqrt{m}}\sqrt{x}\left\{1-\tfrac{\sqrt{x}}{\sqrt{m}}\right\}\\&\ge 0\qquad\qquad \text{for } 0\le x\le m \text{ and } f\ge 0.\end{aligned}$$

Now consider $P_L(x)$ versus $P(x)=(c\,x+f\,y)$. It is clear that $P_L(x)=P(x)=0$ as $y=0$ when $x=0$. Assume $x>0$ so that $y=1$.

$$\begin{aligned}&P(x)-P_L(x)\\&=(c\,x+f)-\tfrac{2f}{\sqrt{m}}\sqrt{x}-\left\{c-\tfrac{f}{m}\right\}x\\&=f-\tfrac{2f}{\sqrt{m}}\sqrt{x}+\tfrac{f}{m}x\\&=f\left\{1-\tfrac{\sqrt{x}}{\sqrt{m}}\right\}^2\\&\ge 0 \text{ and } \le f\quad \text{for } 0\le x\le m \text{ and } f\ge 0.\end{aligned}$$

2.1. P_L approximation for problem P

$$\mathbf{P_L}:\ \text{Minimize}\quad Z=\sum_{i=1}^{S}\sum_{j=1}^{D}\frac{2f_{ij}}{\sqrt{m_{ij}}}\sqrt{x_{ij}}+\left\{c-\frac{f_{ij}}{m_{ij}}\right\}x_{ij}. \tag{8}$$

$$\begin{aligned}\text{Subject to}\quad &\sum_{j=1}^{D}x_{ij}=a_i\quad \text{for } i=1,2,\ldots,S,\\&\sum_{i=1}^{S}x_{ij}=b_j\quad \text{for } j=1,2,\ldots,D,\\&x_{ij}\ge 0\quad \text{for all } (i,j),\end{aligned}$$

The optimal solution $\{x^L_{ij}\}$ to problem $\mathbf{P_L}$ can be easily modified into a feasible solution, $\{x^L_{ij}, y^L_{ij}\}$, of $\mathbf{P}$ as follows:

$$\begin{aligned}&y^L{}_{ij}=0\quad \text{if}\quad x^L{}_{ij}=0,\\ \text{and}\quad &y^L{}_{ij}=1\quad \text{if}\quad x^L{}_{ij}>0.\end{aligned}$$