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# Straight-line drawings of outerplanar graphs in $O(dn \log n)$ area ☆

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ABSTRACT

We show an algorithm for constructing $O(dn \log n)$ area outerplanar straight-line drawings of $n$-vertex outerplanar graphs with degree $d$. Also, we settle in the negative a conjecture (Biedl, 2002 [1]) on the area requirements of outerplanar graphs by showing that snowflake graphs admit linear-area drawings.



## 1. Introduction

Constructing small-area straight-line drawings of planar graphs is a classical research topic in Graph Drawing. Ground-breaking works of the end of the 80's have shown that every planar graph admits a planar straight-line drawing on a grid whose size is quadratic in the number of vertices of the graph [2,3]. It turns out that such an area requirement is worst-case the best possible [4,2,5]. Consequently, the problem of finding non-trivial sub-classes of planar graphs that admit sub-quadratic area drawings has been widely investigated. For example, it is known that every $n$-node tree can be drawn in $O(n \log n)$ area (a simple modification of the *h–v drawing algorithm* in [6]) and that every $n$-node tree whose degree is $O(\sqrt{n})$ can be drawn in $O(n)$ area [7].

One of the classes of graphs that has attracted more research interest is the one of *outerplanar graphs*. An outerplanar graph is a graph that admits a planar drawing in which all vertices are on the same face. Almost thirty years ago, Dolev and Trickey [8] showed that every $n$-vertex outerplanar graph whose degree is bounded by four admits a poly-line drawing in $O(n)$ area. The techniques presented in [8] can be modified in order to obtain poly-line drawings of outerplanar graphs with degree $d$ in $O(d^2 n)$ area, as pointed out in [1]. More recently, the problem of obtaining minimum-area drawings of outerplanar graphs has been tackled by Biedl [1], who first provided a sub-quadratic area upper bound for poly-line drawings of general outerplanar graphs. Namely, she proved that outerplanar graphs admit poly-line drawings in $O(n \log n)$ area. Moreover, she conjectured that there exists a class of outerplanar graphs called "snowflake graphs" requiring $\Omega(n \log n)$ area in any planar straight-line or poly-line drawing.

Concerning straight-line drawings, in [9] (further published in [10]) Garg and Rusu have shown that every $n$-vertex outerplanar graph with degree $d$ has a straight-line drawing with $O(dn^{1.48})$ area. The first sub-quadratic area upper bound has been proved in [11] (further published in [12]), where Di Battista and Frati showed that $O(n^{1.48})$ area always suffices for

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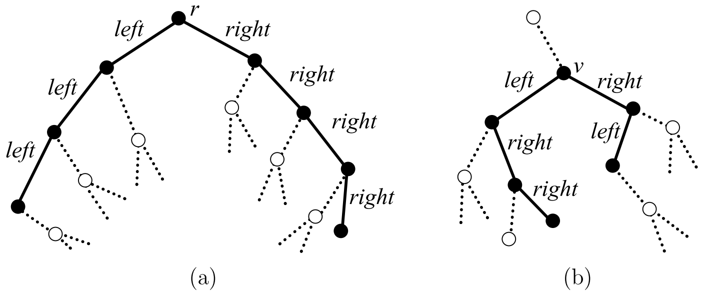


**Fig. 1.** (a) The leftmost and the rightmost path of an ordered rooted binary tree. (b) The left–right path and the right–left path of a node $v$ in an ordered rooted binary tree.

straight-line drawings of outerplanar graphs. For restricted classes of outerplanar graphs, namely outerplanar graphs whose dual tree has a small diameter [1], "complete outerplanar graphs" [12], and "label-constrained outerplanar graphs" [13], better area bounds are known.

In this paper we show an algorithm for obtaining straight-line drawings of outerplanar graphs in $O(dn \log n)$ area. Clearly, this improves the upper bound on the area requirement of all the outerplanar graphs whose degree is $O(n^{0.48}/\log n)$. Further, we prove that snowflake graphs admit linear-area straight-line drawings, settling in the negative the above cited conjecture appeared in [1].

The rest of the paper is organized as follows. Section 2 contains some preliminaries. Section 3 presents an algorithm for constructing straight-line drawings of outerplanar graphs in $O(dn \log n)$ area. Section 4 presents an algorithm for constructing straight-line drawings of snowflake graphs in $O(n)$ area; in the same section we give conclusions and a conjecture concerning the area requirement of straight-line drawings of outerplanar graphs.

## 2. Preliminaries

We assume familiarity with Graph Drawing. For basic definitions see also [14].

A *drawing* of a graph is a mapping of each vertex to a point of the plane and of each edge to a Jordan curve between its endpoints. A drawing is *planar* if no two edges cross, but, possibly, at common endpoints. A *grid drawing* is such that all the vertices have integer coordinates. A *straight-line drawing* (resp. a *poly-line drawing*) is such that each edge is represented by a segment (resp. by a sequence of consecutive segments). Clearly, a straight-line drawing of a graph is fully determined by the placement of its vertices. In the following, unless otherwise specified, for *drawing* we always mean planar straight-line grid drawing. The *area* of a drawing is the number of grid points in the smallest rectangle with sides parallel to the axes that covers the drawing completely.

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same circular ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. Two equivalent drawings of the same graph have the same faces. An embedding of a graph $G$ determines its *dual graph*, that is the graph with one vertex per face of $G$ and with one edge between two vertices if the corresponding faces share an edge in $G$.

An *outerplanar embedding* is a planar embedding in which all the vertices are incident to the same face, say the outer face. *An outerplanar graph* is a graph that admits an outerplanar embedding. The dual graph of an outerplanar embedded graph is a tree when not considering the vertex corresponding to the outer face. A *maximal outerplanar graph* is a graph that admits an outerplanar embedding in which all the faces, except for the outer one, are triangles. The dual graph of a maximal outerplanar embedded graph is a binary tree.

The *degree of a vertex* is the number of edges incident to the vertex. The *degree of a graph* is the maximum degree of one of its vertices. A *binary tree* is a tree such that each node has degree at most three. A *rooted tree* is one with a distinguished node, called *root*. A tree is *ordered* if a left-to-right order of the children of each node is fixed. A drawing $\Gamma$ of an ordered tree $T$ is *order-preserving* if the order in which the edges are incident to each node in $\Gamma$ is the same as specified in $T$.

Let $T$ be an ordered binary tree rooted at node $r$. Let $T(v)$ denote the subtree of $T$ rooted at node $v$. The *leftmost path* $L(T)$ (resp. the *rightmost path* $R(T)$) of $T$ is the path $v_0, v_1, \dots, v_k$ such that $v_0 = r$, $v_{i+1}$ is the left child (resp. the right child) of $v_i$, $\forall i$: $0 \leqslant i \leqslant k-1$, and $v_k$ does not have a left child (resp. a right child). See Fig. 1(a).

The *left–right path* (resp. the *right–left path*) of a node $v \in T$ is the path $v_0, v_1, \dots, v_k$ such that $v_0 = v$, $v_1$ is the left (resp. right) child of $v_0$, and $v_1, v_2, \dots, v_k$ is the rightmost (resp. leftmost) path of $T(v_1)$. See Fig. 1(b).

Consider any drawing $\Gamma$ of $T$. The *left polygon of the neighbors* $P_l(v)$ (resp. *the right polygon of the neighbors* $P_r(v)$) of a node $v \in T$ is the polygon of the segments representing in $\Gamma$ the edges of the left–right path (resp. of the right–left path) plus a segment connecting $v_k$ and $v_0$.