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Relaxing the constraints of clustered planarity

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ABSTRACT

In a drawing of a clustered graph vertices and edges are drawn as points and curves, respectively, while clusters are represented by simple closed regions. A drawing of a clustered graph is c-planar if it has no edge–edge, edge–region, or region–region crossings. Determining the complexity of testing whether a clustered graph admits a c-planar drawing is a long-standing open problem in the Graph Drawing research area. An obvious necessary condition for c-planarity is the planarity of the graph underlying the clustered graph. However, this condition is not sufficient and the consequences on the problem due to the requirement of not having edge–region and region–region crossings are not yet fully understood.

In order to shed light on the c-planarity problem, we consider a relaxed version of it, where some kinds of crossings (either edge–edge, edge–region, or region–region) are allowed even if the underlying graph is planar. We investigate the relationships among the minimum number of edge–edge, edge–region, and region–region crossings for drawings of the same clustered graph. Also, we consider drawings in which only crossings of one kind are admitted. In this setting, we prove that drawings with only edge–edge or with only edge–region crossings always exist, while drawings with only region–region crossings may not. Further, we provide upper and lower bounds for the number of such crossings. Finally, we give a polynomial-time algorithm to test whether a drawing with only region–region crossings exists for biconnected graphs, hence identifying a first non-trivial necessary condition for c-planarity that can be tested in polynomial time for a noticeable class of graphs.

1. Introduction

Clustered planarity is a classical Graph Drawing topic (see [6] for a survey). A *clustered graph* $C(G,T)$ consists of a graph G and of a rooted tree T whose leaves are the vertices of G. Such a structure is used to enrich the vertices of the graph with hierarchical information. In fact, each internal node μ of T represents the subset, called *cluster*, of the vertices of G that are the leaves of the subtree of T rooted at μ. Tree T, which defines the inclusion relationships among clusters, is called *inclusion tree*, while G is the *underlying graph* of $C(G,T)$.

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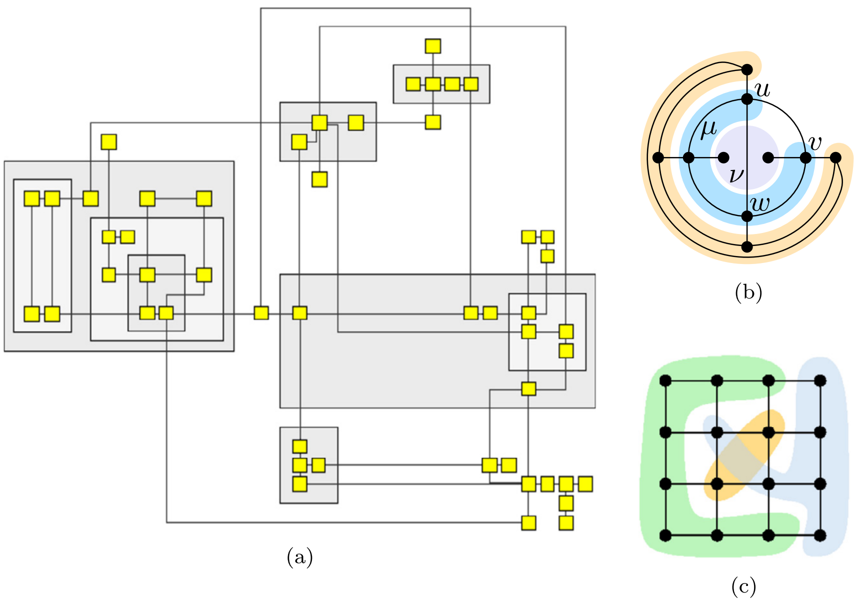

Fig. 1. Examples of crossings in drawings of clustered graphs. (a) A drawing obtained with the planarization algorithm described in [11] and containing three edge–edge crossings. (b) A drawing with two edge–region crossings. (c) A drawing with a region–region crossing.

In a *drawing* of a clustered graph $C(G, T)$ vertices and edges of G are drawn as points and open curves, respectively, and each node μ of T is represented by a simple closed region $R(\mu)$ containing exactly the vertices of μ. Also, if μ is a descendant of a node ν, then $R(\nu)$ contains $R(\mu)$.

A drawing of C can have three types of crossings. *Edge–edge crossings* are crossings between edges of G. Algorithms to produce drawings allowing edge–edge crossings have already been proposed (see, for example, [11] and Fig. 1(a)). Two kinds of crossings involve regions, instead. Consider an edge e of G and a node μ of T. If e intersects the boundary of $R(\mu)$ only once, this is not considered as a crossing since there is no way of connecting the endpoints of e without intersecting the boundary of $R(\mu)$. On the contrary, if e intersects the boundary of $R(\mu)$ more than once, we have *edge–region crossings*. An example of this kind of crossings is provided by Fig. 1(b), where edge (u, w) traverses $R(\mu)$ and edge (u, v) exits and enters $R(\nu)$. Finally, consider two nodes μ and ν of T; if the boundary of $R(\mu)$ intersects the boundary of $R(\nu)$, we have a *region–region crossing* (see Fig. 1(c) for an example).

A drawing of a clustered graph is *c-planar* if it does not have any edge–edge, edge–region, or region–region crossing. A clustered graph is *c-planar* if it admits a c-planar drawing.

In the last decades c-planarity has been deeply studied. While the complexity of deciding if a clustered graph is c-planar is still an open problem in the general case, polynomial-time algorithms have been proposed to test c-planarity and produce c-planar drawings under several kinds of restrictions, such as:

- Assuming that each cluster induces a small number of connected components [5,7,10,16,17,21,22,24,25]. In particular, the case in which the graph is *c-connected*, that is, for each node ν of T the graph induced by the vertices of ν is connected, has been deeply investigated.
- Considering only *flat* hierarchies, i.e., the height of T is two, namely no cluster different from the root contains other clusters [8,9,12].
- Focusing on particular families of underlying graphs [8,9,26].
- Fixing the embedding of the underlying graph [12,24].

This huge body of research can be read as a collection of polynomial-time testable sufficient conditions for c-planarity.

In contrast, the planarity of the underlying graph is the only polynomial-time testable necessary condition that has been found so far for c-planarity in the general case. Such a condition, however, is not sufficient and the consequences on the problem due to the requirement of not having edge–region and region–region crossings are not yet fully understood.

Other known necessary conditions are either trivial (i.e., satisfied by all clustered graphs) or of unknown complexity as the original problem is. An example of the first kind is the existence of a c-planar clustered graph obtained by splitting some cluster into sibling clusters [2]. An example of the second kind, which is also a sufficient condition, is the existence of a set of edges that, if added to the underlying graph, make the clustered graph c-connected and c-planar [16].

In this paper we study a relaxed model of c-planarity. Namely, we study $\langle\alpha, \beta, \gamma\rangle$-drawings of clustered graphs. In an $\langle\alpha, \beta, \gamma\rangle$-drawing the number of edge–edge, edge–region, and region–region crossings is equal to α, β, and γ, respectively. Figs. 1(a), 1(b), and 1(c) show examples of a $\langle 3, 0, 0\rangle$-drawing, a $\langle 0, 2, 0\rangle$-drawing, and a $\langle 0, 0, 1\rangle$-drawing, respectively.