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Seven mutually touching infinite cylinders

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ABSTRACT

We solve a problem of Littlewood: there exist seven infinite circular cylinders of unit radius which mutually touch each other. In fact, we exhibit two such sets of cylinders. Our approach is algebraic and uses symbolic and numerical computational techniques. We consider a system of polynomial equations describing the position of the axes of the cylinders in the 3 dimensional space. To have the same number of equations (namely 20) as the number of variables, the angle of the first two cylinders is fixed to 90 degrees, and a small family of direction vectors is left out of consideration. Homotopy continuation method has been applied to solve the system. The number of paths is about 121 billion, it is hopeless to follow them all. However, after checking 80 million paths, two solutions are found. Their validity, i.e., the existence of exact real solutions close to the approximate solutions at hand, was verified with the alphaCertified method as well as by the interval Krawczyk method.

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1. Littlewood's problem on seven touching infinite cylinders

John Edensor Littlewood ([11], Problem 7 on page 20) proposed that

"*Is it possible in 3-space for seven infinite circular cylinders of unit radius each to touch all the others? Seven is the number suggested by constants.*"

Two cylinders touch each other if their intersection is either a point or a line. Ogilvy's book [12] also includes Littlewood's problem.

Finite versions of the problem are discussed as puzzles by Gardner and they are well-known as 6 touching cigarettes [5, Fig. 54 on page 115] and 7 touching cigarettes [5, Fig. 55 on page 115]. The latter works for a ratio of length/diameter

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greater than $7\sqrt{3}/2$. However, as it is noted by Bezdek [2], it is still open whether it is possible to find 8 or more touching finite identical cylinders. An arrangement of 5 touching coins (with a *small* ratio of length/diameter) is also known [5, Fig. 49 on page 110] and this fact suggests that intermediate ratios of length/diameter could also be analyzed.

Bezdek [2] showed that 24 is an upper bound for the number of mutually touching congruent infinite cylinders. Ambrus and Bezdek [1] investigated the proposal of Kuperberg from the early 1990's that contained 8 congruent infinite cylinders. It is shown that they do not mutually touch each other, see [1, Theorem 1 and Fig. 1 on page 1804] for details. Brass, Moser and Pach discuss an arrangement of 6 mutually touching infinite cylinders [3, page 98]. In the paper this lower bound is improved to 7.

Hereafter, it is assumed that cylinders are infinite and congruent, their radius is set to 1. Two cylinders of unit radius touch each other if and only if the distance of their axes is 2. Let C_i and ℓ_i denote the i-th cylinder and its axis, respectively. In the paper, $i = 1, 2, \dots, 7$. The case of parallel cylinders (lines) is excluded from our analysis. It is left to the reader to show that if two cylinders are parallel, then the maximum number of mutually touching cylinders is four.

We intend to apply the well-known formula for the distance of two lines in $\mathbb{R}^3$. Let

$$\ell_i(s) = \mathbf{P}_i + s\,\mathbf{w}_i$$

be a parametric representation of line ℓ_i for $i = 1, \dots, 7$. Here $\mathbf{P}_i \in \mathbb{R}^3$ is a point of ℓ_i, $\mathbf{w}_i \in \mathbb{R}^3$ is a direction vector and s is a real parameter. If lines ℓ_i and ℓ_j are skew, then their distance can be obtained as

$$d(\ell_i, \ell_j) = \frac{|(\overrightarrow{\mathbf{P}_i\mathbf{P}_j}) \cdot (\mathbf{w}_i \times \mathbf{w}_j)|}{\|\mathbf{w}_i \times \mathbf{w}_j\|}, \tag{1}$$

where $\cdot$ denotes dot product, $\times$ denotes cross product and $\|\ \|$ denotes the Euclidean norm [6,20]. Since the cylinders have unit radius, $d(\ell_i, \ell_j) = 2$ for all $i, j = 1, 2, \dots, 7$, $i \neq j$, we can write Eq. (1) as

$$\left|(\overrightarrow{\mathbf{P}_i\mathbf{P}_j}) \cdot (\mathbf{w}_i \times \mathbf{w}_j)\right|^2 - 4\|\mathbf{w}_i \times \mathbf{w}_j\|^2 = 0. \tag{2}$$

In this form we avoid taking square roots. Let us introduce coordinates:

$$\mathbf{P}_i = (x_i, y_i, z_i), \qquad \mathbf{w}_i = (t_i, u_i, v_i).$$

Then we have

$$\overrightarrow{\mathbf{P}_i\mathbf{P}_j} = (x_j - x_i, y_j - y_i, z_j - z_i), \tag{3}$$

$$\mathbf{w}_i \times \mathbf{w}_j = (u_i v_j - v_i u_j, v_i t_j - t_i v_j, t_i u_j - u_i t_j). \tag{4}$$

Now we substitute (3)–(4) into (2), and by using the well-known determinantal form of the triple product, we obtain the equation

$$\det \begin{bmatrix} x_j - x_i & y_j - y_i & z_j - z_i \\ t_i & u_i & v_i \\ t_j & u_j & v_j \end{bmatrix}^2 - 4\left((u_i v_j - v_i u_j)^2 + (v_i t_j - t_i v_j)^2 + (t_i u_j - u_i t_j)^2\right) = 0. \tag{5}$$

This is a polynomial equation of degree 6 in 12 variables. The polynomial on the left is a linear combination of 84 monomials.

We call a line horizontal if it is parallel to the plane $z = 0$. Any arrangement of seven lines can be translated and rotated to a position in which one of the lines (ℓ_1) is horizontal, with direction vector $\mathbf{w}_1 = (1, 0, 0)$, and it goes through the point $\mathbf{P}_1(0, 0, -1)$. It can also be assumed that the touching point of cylinders C_1 and C_2 is $(0, 0, 0)$, that is, ℓ_2 goes through the point $\mathbf{P}_2(0, 0, 1)$. The direction of (ℓ_2) is the only degree of freedom when the first two lines are considered. We shall assume, and this is explained later, that (ℓ_2) will be chosen to be orthogonal to the first line. We have so far

$$x_1 = 0, \quad y_1 = 0, \quad z_1 = -1, \quad t_1 = 1, \quad u_1 = 0, \quad v_1 = 0; \tag{6}$$

$$x_2 = 0, \quad y_2 = 0, \quad z_2 = 1, \quad t_2 = 0, \quad u_2 = 1, \quad v_2 = 0. \tag{7}$$

We can make some further simplifications. We may assume without loss of generality that ℓ_i $(i = 3, \dots, 7)$ is not horizontal (otherwise it would be parallel to ℓ_1 or ℓ_2), consequently, it goes through the plane $z = k$ for any $k \in \mathbb{R}$. Let us choose $k = 0$ and set

$$z_i = 0 \quad \text{for } i = 3, \dots, 7. \tag{8}$$

Finally, the normalization of the direction vector of line ℓ_i is chosen to be $t_i + u_i + v_i = 1$ for $i = 3, \dots, 7$. This is equivalent to

$$v_i = 1 - t_i - u_i, \quad i = 3, \dots, 7. \tag{9}$$

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