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Weighted straight skeletons in the plane ☆

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ABSTRACT

We investigate weighted straight skeletons from a geometric, graph-theoretical, and combinatorial point of view. We start with a thorough definition and shed light on some ambiguity issues in the procedural definition. We investigate the geometry, combinatorics, and topology of faces and the roof model, and we discuss in which cases a weighted straight skeleton is connected. Finally, we show that the weighted straight skeleton of even a simple polygon may be non-planar and may contain cycles, and we discuss under which restrictions on the weights and/or the input polygon the weighted straight skeleton still behaves similar to its unweighted counterpart. In particular, we obtain a non-procedural description and a linear-time construction algorithm for the straight skeleton of strictly convex polygons with arbitrary weights.

1. Introduction

The straight-skeleton $\mathcal{S}(P)$ of a simple polygon P is a skeleton structure that was introduced to computational geometry by Aichholzer et al. [1] about 20 years ago. Its definition is based on a wavefront propagation process where the polygon's edges move inwards at unit speed. The straight skeleton, roughly speaking, is the skeleton structure that results from the interference patterns of the wavefront edges. Aichholzer and Aurenhammer [2] later generalized the definition to planar straight-line graphs. Since their introduction, a lot of applications appeared in different research areas, and multiple algorithms to compute the straight skeleton have been introduced [3].

Eppstein and Erickson [4] were the first to mention the *weighted straight skeleton* where the wavefront edges may move with arbitrary but fixed speeds. They claim that their algorithm to compute the unweighted straight skeleton in $O(n^{8/5+\epsilon})$ time and space also works, without major changes, for weighted straight skeletons. Weighted straight skeletons have many applications: Barequet et al. [5] use weighted straight skeletons in order to define the initial wavefront topology for straight skeletons of polyhedra. Haunert and Sester [6] use the weighted straight skeleton for topology-preserving area collapsing in geographic maps. Laycock and Day [7] and Kelly and Wonka [8] use weighted straight skeletons to model realistic roofs of houses. Aurenhammer [9] investigated fixed-share decompositions of convex polygons using weighted straight skeletons

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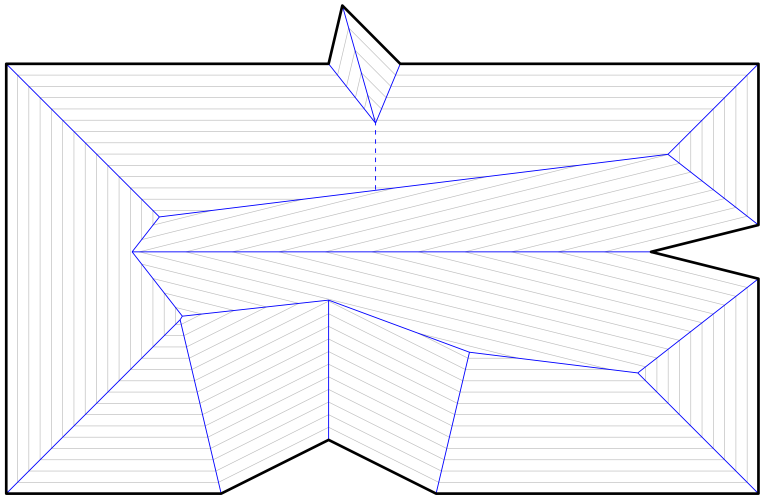

Fig. 1. The straight skeleton $\mathcal{S}(P)$ of the input polygon P (bold) is defined by wavefronts emanated from P. The dashed arc is traced by a ghost vertex.

with specific positive weights. Sugihara [10] employs weighted straight skeletons for the interactive design of pop-up cards.

Although algorithms, applications, and even simple implementations [11] of weighted straight skeletons are known, only limited research has been conducted on the weighted straight skeleton itself. The only known results are that the simple definition based on wavefront propagation may lead to ambiguities [3,8] and that the lower envelope characterization by Eppstein and Erickson [4] does not apply. In this paper, we carefully define weighted straight skeletons, shed light on the ambiguity in the procedural definition, investigate geometric, graph-theoretical and combinatorial properties of weighted straight skeletons, and compare those with properties of unweighted straight skeletons. In particular, we show that weighted straight skeletons of simple polygons may have cycles and crossings. Furthermore, we investigate necessary conditions for the weights or the polygon such that the weighted straight skeleton of a simple polygon is a planar tree.

2. Preliminaries

The definition of the straight skeleton $\mathcal{S}(P)$ of a simple polygon P is based on a so-called wavefront propagation of P where all edges of P move inwards in parallel and at unit speed. (This definition is readily extended to polygons with holes.) The wavefront, denoted by $\mathcal{W}_P(t)$, has the shape of a mitered offset curve of P for small t. As t increases, $\mathcal{W}_P$ changes its topology. Such changes are called events and we can distinguish between two main types: An *edge event* happens when an edge e collapses to zero length and vanishes. A *split event* happens when a reflex wavefront vertex v reaches another part of the wavefront and splits it into parts. Typically, v meets a wavefront edge whose split causes the entire wavefront to split into two parts. However, if v meets one or more other wavefront vertices, then more complicated splits of the wavefront into multiple parts are possible. Either event causes local changes in the topology of the wavefront so that the resulting wavefront again consists of a collection of simple polygons.

The straight skeleton $\mathcal{S}(P)$ is defined as the set of loci traced out by the vertices of $\mathcal{W}_P(t)$ for all $t \geq 0$, see Fig. 1. Additionally, some loci are added to the straight skeleton in case of parallel edges as follows: (a) If two parallel edges e and e' that move in opposite directions become overlapping during an event, then the region common to e and e' is added to the straight skeleton, while the region(s) that belongs to exactly one of them remains in the wavefront. (b) If two parallel edges e and e' that move in the same direction become adjacent due to an edge event, then their common endpoint is considered a vertex of the wavefront. We call this a *ghost vertex*. This vertex moves perpendicular to e and e'.[1]

Each wavefront vertex traces out an *arc* of $\mathcal{S}(P)$. Since wavefront vertices move along bisectors of edges of P, the arcs of $\mathcal{S}(P)$ are straight-line segments. Every event of $\mathcal{W}_P$ corresponds to a locus where arcs of $\mathcal{S}(P)$ meet and give rise to a *node* of $\mathcal{S}(P)$. See Fig. 1 for an example. The straight skeleton $\mathcal{S}(P)$ is interpreted as a graph, and one can show that it is a tree [1]. Also, no two arcs of the straight skeleton cross since the wavefront moves inwards towards the unswept region. Hence, $\mathcal{S}(P) \cup P$ is a planar straight-line graph. The inner faces of $\mathcal{S}(P) \cup P$ are called straight-skeleton *faces*.

Let e be an edge of a polygon P, and let the *wavefront fragments of e at time t* be the union of segments of $\mathcal{W}_P(t)$ that originated from e. We denote this set by $e(t)$, and it may comprise none, one, or many segments, depending on whether e participated in edge events and/or split events. We consider segments in $e(t)$ to be *open* line segments. Every

[1] One could also argue for omitting ghost vertices, thus effectively merging e and e' into one edge of the wavefront. But then, even in the unweighted case, the straight skeleton is not always connected.