Computational Geometry 48 (2015) 415–428

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Reprint of: Face-guarding polyhedra

Giovanni Viglietta

School of Electrical Engineering and Computer Science, University of Ottawa, Ottawa ON, Canada

ARTICLE INFO

Article history:
Available online 12 January 2015

Keywords:
Art Gallery
Face guards
Polyhedra
Terrains

ABSTRACT

We study the Art Gallery Problem for face guards in polyhedral environments. The problem can be informally stated as: *how many (not necessarily convex) windows should we place on the external walls of a dark building, in order to completely illuminate its interior?*

We consider both *closed* and *open* face guards (i.e., faces with or without their boundary), and we study several classes of polyhedra, including *orthogonal* polyhedra, *4-oriented* polyhedra, and *2-reflex orthostacks*.

We give upper and lower bounds on the minimum number of faces required to guard the interior of a given polyhedron in each of these classes, in terms of the total number of its faces, f. In several cases our bounds are tight: $\lfloor f/6 \rfloor$ *open* face guards for orthogonal polyhedra and 2-reflex orthostacks, and $\lfloor f/4 \rfloor$ *open* face guards for 4-oriented polyhedra. Additionally, for *closed* face guards in 2-reflex orthostacks, we give a lower bound of $\lfloor (f+3)/9 \rfloor$ and an upper bound of $\lfloor (f+1)/7 \rfloor$.

Then we show that it is **NP**-hard to approximate the minimum number of (closed or open) face guards within a factor of $\Omega(\log f)$, even for polyhedra that are orthogonal and simply connected. We also obtain the same hardness results for *polyhedral terrains*.

Along the way we discuss some applications, arguing that face guards are *not* a reasonable model for guards *patrolling* on the surface of a polyhedron.

1. Introduction

Previous work. Art Gallery Problems have been studied in computational geometry for decades: given an *enclosure*, place a (preferably small) set of *guards* such that every location in the enclosure is seen by some guard. Most of the early research on the Art Gallery Problem focused on guarding 2-dimensional polygons with either point guards or segment guards [12,13,15].

Gradually, some of the attention started shifting to 3-dimensional settings, as well. Several authors have considered edge guards in 3-dimensional polyhedra, either in relation to the classical Art Gallery Problem or to its variations [3,5,6,16,17].

Recently, Souvaine et al. [14] introduced the model with *face guards* in 3-dimensional polyhedra. Ideally, each guard is free to roam over an entire face of a polyhedron, including the face's boundary. Let $g(\mathcal{P})$ be the minimum number of face guards needed for a polyhedron $\mathcal{P}$, and let $g(f)$ be the maximum of $g(\mathcal{P})$ over all polyhedra $\mathcal{P}$ with exactly f faces. For general polyhedra, Souvaine et al. showed that $\lfloor f/5 \rfloor \leqslant g(f) \leqslant \lfloor f/2 \rfloor$ and, for the special case of orthogonal polyhedra (i.e., polyhedra whose faces are orthogonal to the coordinate axes), they showed that $\lfloor f/7 \rfloor \leqslant g(f) \leqslant \lfloor f/6 \rfloor$. They also

DOI of original article: http://dx.doi.org/10.1016/j.comgeo.2014.04.009. The publisher would like to inform the readership that this article is a reprint of a previously published article. An error occurred which resulted in the publication of this article in a wrong issue. As a consequence, the publisher would like to make this reprint available for the reader's convenience and for the continuity of the papers involved in the Special Issue.

E-mail address: viglietta@gmail.com.

http://dx.doi.org/10.1016/j.comgeo.2015.01.003
0925-7721/© 2015 Published by Elsevier B.V.

suggested several open problems, such as studying *open* face guards (i.e., face guards whose boundary is omitted), and the computational complexity of minimizing the number of face guards.

Subsequently, face guards have been studied to some extent also in the case of polyhedral terrains. In [9,11] a tight bound is obtained, and in [10] it is proven that minimizing face guards in triangulated terrains is **NP**-hard. However, since these results apply to terrains, they have no direct implications on the problem of face-guarding polyhedral enclosures.

Our contribution. In this paper we solve some of the problems left open in [14], and we also expand our research in some new directions. A preliminary version of this paper has appeared at CCCG 2013 [18].

In Section 2 we discuss the face guard model, arguing that a face guard fails to meaningfully represent a guard "patrolling" on a face of a polyhedron. Essentially, there are cases in which the path that such a patrolling guard ought to follow is so complex (in terms of the number of turns, if it is a polygonal chain) that a much simpler path, striving from the face, would guard not only the region visible from that face, but the entire polyhedron. However, face guards are still a good model for illumination-related problems, such as placing (possibly non-convex) windows in a dark building.

In Section 3 we obtain some new bounds on $g(f)$, for both closed and open face guards. First we generalize the upper bounds given in [14] by showing that, for c-oriented polyhedra (i.e., whose faces have c distinct orientations), $g(f) \leqslant \lfloor f/2 - f/c \rfloor$. We also provide some new lower bound constructions, which meet our upper bounds in two notable cases: orthogonal polyhedra with open face guards ($g(f) = \lfloor f/6 \rfloor$), and 4-oriented polyhedra with open face guards ($g(f) = \lfloor f/4 \rfloor$). Then we go on to study a special class of orthogonal polyhedra, namely *2-reflex orthostacks*.

The following table summarizes our new results, as well as those that were already known. Each entry contains a lower and an upper bound on $g(f)$, or a single tight bound. When applicable, a reference is given to the paper in which each result was first obtained. Observe that, for open face guards in triangulated terrains, f guards are easily seen to be necessary in the worst case. Indeed, if the terrain is a convex "dome" (i.e., if no edges are reflex), then every face requires an open face guard. In the case of closed face guards in triangulated terrains, we remark that the bound given in [11] is expressed in terms of the number of vertices. Therefore we rewrote it in terms of f, using Euler's formula.

	Open face guards	Closed face guards
2-Reflex orthostacks	$g(f) = \lfloor f/6 \rfloor$	$\lfloor (f+3)/9 \rfloor \leqslant g(f) \leqslant \lfloor (f+1)/7 \rfloor$
Orthogonal polyhedra	$g(f) = \lfloor f/6 \rfloor$	$\lfloor f/7 \rfloor \leqslant_{[14]} g(f) \leqslant_{[14]} \lfloor f/6 \rfloor$
4-Oriented polyhedra	$g(f) = \lfloor f/4 \rfloor$	$\lfloor f/5 \rfloor \leqslant g(f) \leqslant \lfloor f/4 \rfloor$
General polyhedra	$\lfloor f/4 \rfloor \leqslant g(f) \leqslant \lfloor f/2 \rfloor - 1$	$\lfloor f/5 \rfloor \leqslant_{[14]} g(f) \leqslant \lfloor f/2 \rfloor - 1$
Triangulated terrains	$g(f) = f$	$g(f) =_{[11]} \lfloor (f+3)/6 \rfloor$

In Section 4 we provide an approximation-preserving reduction from Set Cover to the problem of minimizing the number of (closed or open) face guards in simply connected orthogonal polyhedra. It follows that the minimum number of face guards is **NP**-hard to approximate within a factor of $\Omega(\log f)$. We also obtain the same result for (non-triangulated) terrains. This adds to the result of [10], which states that minimizing closed face guards is **NP**-hard in triangulated terrains. We also briefly discuss the membership in **NP** of the minimization problem, pointing out some difficulties in applying previously known techniques.

We leave as an open problem the task to tighten all the bounds in the table above, as well as to prove or disprove that minimizing face guards is in **NP**. We conjecture that all the lower bounds are tight, and that the minimization problem does belong to **NP**.

2. Model and motivations

Definitions. A *polyhedron* is a connected subset of $\mathbb{R}^3$, union of finitely many closed tetrahedra embedded in $\mathbb{R}^3$, whose boundary is a (possibly non-connected) orientable 2-manifold. Since a polyhedron's boundary is piecewise linear, the notion of *face* of a polyhedron is well defined as a maximal planar subset of its boundary with connected and non-empty relative interior. Thus a face is a plane polygon, possibly with holes, and possibly with some degeneracies, such as hole boundaries touching each other at a single vertex. Any vertex of a face is also considered a *vertex* of the polyhedron. *Edges* are defined as minimal non-degenerate straight line segments shared by two distinct faces and connecting two vertices of the polyhedron. Since a polyhedron's boundary is an orientable 2-manifold, the relative interior of an edge lies on the boundary of exactly two faces, thus determining an internal dihedral angle (with respect to the polyhedron). An edge is *reflex* if its internal dihedral angle is reflex, i.e., strictly greater than 180°.

Given a polyhedron, we say that a point x is *visible* to a point y if no point in the straight line segment xy lies in the exterior of the polyhedron. For any point x, we denote by $\mathcal{V}(x)$ the *visible region* of x, i.e., the set of points that are visible to x. In general, for any set $S \subset \mathbb{R}^3$, we let $\mathcal{V}(S) = \bigcup_{x \in S} \mathcal{V}(x)$. A set is said to *guard* a polyhedron if its visible region coincides with the entire polyhedron (including its boundary). The Art Gallery Problem for face guards in polyhedra consists in finding a (preferably small) set of faces whose union guards a given polyhedron. If such faces include their relative boundary, they are called *closed* face guards; if their boundary is omitted, they are called *open* face guards.

A polyhedron is *c-oriented* if there exist c unit vectors such that each face is orthogonal to one of the vectors. If these unit vectors form an orthonormal basis of $\mathbb{R}^3$, the polyhedron is said to be *orthogonal*. Hence, a cube is orthogonal, a tetrahedron and a regular octahedron are both 4-oriented, etc.