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A tight bound for point guards in piecewise convex art galleries

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ABSTRACT

We consider the problem of guarding curvilinear art galleries. A Jordan arc a joining two points, p and q, in the plane is called a *convex arc* if the closed curve obtained by joining a with the line segment pq encloses a convex set. A piecewise convex art gallery A with n vertices is a simply connected region in the plane whose boundary consists of n convex arcs where A lies on the convex side of each arc. We show that $\lceil \frac{n}{2} \rceil$ point guards are always sufficient and sometimes necessary to guard a piecewise convex art gallery with n vertices. We also give a shorter proof for the sufficiency of $\lfloor \frac{2n}{3} \rfloor$ vertex guards, for $n \geqslant 2$, which was first derived by Karavelas et al.

1. Introduction

In a typical art gallery problem, we are given a closed set A in the plane (the floorplan of a gallery), and would like to find a minimum set $G \subset A$ (*guards*) that jointly monitor A. A set $G \subseteq A$ of guards jointly monitor A if for any point $p \in A$ there is a point $q \in G$ such that the line segment pq lies in A. In the 1970s, Chvátal [6] proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to monitor a simple polygonal domain with n vertices. However, it is NP-hard to decide whether k guards are sufficient to monitor a given simple polygonal domain [14]. Since then, many variations of this problem have been studied; see [20,19,23] for a detailed reference on art gallery problems. Areas of applications include robotics [13,24], motion planning [15,18], computer vision and pattern recognition [2,21,22,25], computer graphics [5,16], CAD/CAM [3,9], and wireless networks [10].

Recently Karavelas, Tsigaridas, and Tóth [12] generalized the art gallery problem to curvilinear art galleries. A region A in the plane is a *curvilinear art gallery* with n vertices if A is simply connected and its boundary contains n distinct points $\{v_0, v_1, \ldots, v_{n-1}\}$, which are the vertices of A. The *edges* of A are the connected Jordan arcs of the boundary of A between consecutive vertices. Specifically, the arc a_i is the part of the boundary of A from v_i to $v_{i+1 \bmod n}$ in counterclockwise order. We study curvilinear galleries with well-behaved edges. A Jordan arc a between points p and q is *convex* if the closed curve containing a and the line segment qp encloses a convex region of the plane. A curvilinear art gallery in which all edges are convex arcs is called a *splinegon*, introduced by Dobkin and Souvaine [7]. In general, the minimum number of guards for a curvilinear art gallery or for a splinegon cannot be bounded in terms of the number of vertices [12].

Notice that each convex arc has a convex and a concave side. A curvilinear art gallery is *piecewise convex* if its edges are convex arcs and its interior lies on the convex side of every edge (Fig. 1, left). Our main result (Theorem 1) shows that every

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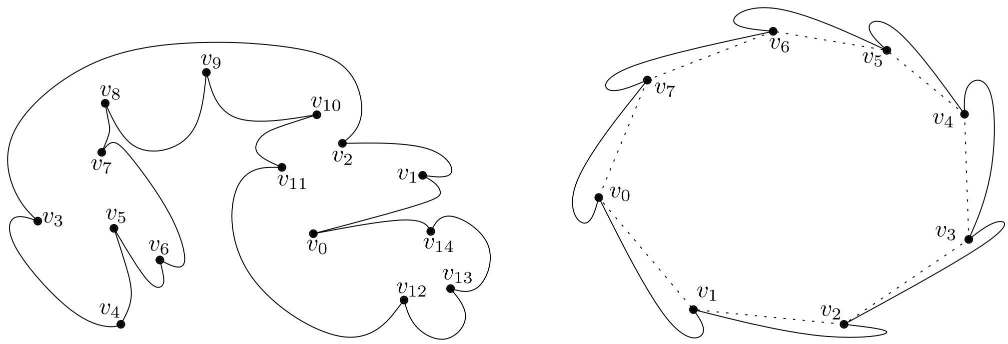

Fig. 1. Left: A piecewise convex art gallery with $n = 15$ vertices. Right: A piecewise convex art gallery with $n = 8$ vertices that requires $\lceil \frac{n}{2} \rceil = 4$ point guards.

piecewise convex art gallery with n vertices can be guarded with at most $\lceil \frac{n}{2} \rceil$ point guards. Piecewise convex art galleries are not only interesting on their own right, but they also lead to tighter upper bounds for certain polygonal art galleries. A *convex chain* of a simple polygon P is the union of consecutive edges of P that are in convex position and appear in the same cyclic order along the convex hull of P. Every simple polygon P can be decomposed into convex chains, which are in fact convex arcs. This gives a representation of a simple polygonal domain as a piecewise convex curvilinear art gallery. This may reduce the number of vertices significantly, and provide a better upper bound for the minimum number of guards.

Karavelas et al. [12] proved that $\lfloor \frac{2n}{3} \rfloor$ vertex guards (that is, guards restricted to be on the vertices of A) are always sufficient and sometimes necessary to guard a piecewise convex art gallery with $n \geqslant 2$ vertices. They also proved that $\lceil \frac{n}{2} \rceil$ point guards (that is, guards positioned anywhere in A) are sometimes necessary. Cano, Espinosa and Urrutia [4] proved that $\lfloor \frac{5n}{8} \rfloor$ point guards are always sufficient to guard a piecewise convex art gallery with $n \geqslant 2$ vertices. Recently, Karavelas [11] showed that $\lfloor \frac{2n+1}{5} \rfloor$ edge guards (that is, guards allowed to move along edges of the gallery) are always sufficient to guard any curvilinear art gallery with n vertices, and $\lfloor \frac{n}{3} \rfloor$ edge guards are sometimes necessary. Our main result is the following.

Theorem 1. *Every piecewise convex art gallery with $n \geqslant 1$ vertices can be monitored by at most $\lceil \frac{n}{2} \rceil$ point guards.*

The upper bound of $\lceil \frac{n}{2} \rceil$ in Theorem 1 is the best possible. For every $n \geqslant 1$, there is a piecewise convex art gallery with n vertices that requires $\lceil \frac{n}{2} \rceil$ point guards. A construction due to Karavelas et al. [12] is shown on the right of Fig. 1.

Proof technique. Our proof is based on a convex decomposition of a piecewise convex art gallery with n vertices into at most $n+1$ convex cells. We partition the set of convex cells into at most $\lceil \frac{n}{2} \rceil$ subsets, each of which can be monitored by a single guard lying on their common boundary. This partition is equivalent to a clique cover of the dual graph of the convex decomposition (defined below). In Section 2, we define several classes of convex decompositions for curvilinear polygons, including the special class where the boundary of every convex cell contains at least two vertices of the gallery. In Section 3, we construct a convex decomposition with the above property for a piecewise convex art gallery by incremental algorithm reminiscent of that of Al-Jubeh et al. [1]. We study the dual graph of the convex decomposition in Section 4, and show that certain subgraphs are Hamiltonian. In Section 5, our convex decomposition is used for placing guards on the boundaries of the convex cells, effectively building a clique cover of the dual graph of the decomposition.

Our convex decomposition is different from previous decomposition methods developed for simple splinegons. Dobkin et al. [8] designed an algorithm for decomposing a monotone piecewise convex art gallery into the *minimum number* of convex cells. Our convex decompositions do not necessarily have the minimum number of cells, but every cell is incident to at least two vertices of the gallery, which helps keeping track of the number of point guards. Melissaratos and Souvaine [17] designed data structures for ray shooting and visibility regions in simple splinegons, based on a cell decomposition whose dual graph has bounded degree. The dual graphs of our decompositions, however, may have arbitrarily high degree.

A simpler proof for a tight bound on vertex guards. To close our paper in Section 6, we give a simpler proof for the sufficiency of $\lfloor \frac{2n}{3} \rfloor$ vertex guards for a curvilinear art gallery with $n \geqslant 2$ vertices, which was first derived by Karavelas et al. [12]. The new proof is a combination of a convex decomposition of the piecewise convex art gallery and a vertex coloring scheme. Our results demonstrate that convex decompositions remain a powerful technical tool for combinatorial results in curvilinear polygons.

2. Convex decompositions

Let A be a curvilinear art gallery with vertices $v_0, v_1, \ldots, v_{n-1}$ and edges $a_0, a_1, \ldots, a_{n-1}$ where a_i is a Jordan arc joining v_i and $v_{i+1 \bmod n}$. A *convex decomposition* of A is a finite set C of closed convex regions, called *cells*, with pairwise disjoint and nonempty interiors such that their union is A. See Fig. 2(left) for an example.