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Refold rigidity of convex polyhedra ☆

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ABSTRACT

We show that every convex polyhedron may be unfolded to one planar piece, and then refolded to a different convex polyhedron. If the unfolding is restricted to cut only edges of the polyhedron, we identify several polyhedra that are "edge-refold rigid" in the sense that each of their unfoldings may only fold back to the original. For example, each of the 43,380 edge unfoldings of a dodecahedron may only fold back to the dodecahedron, and we establish that 11 of the 13 Archimedean solids are also edge-refold rigid. We begin the exploration of which classes of polyhedra are and are not edge-refold rigid, demonstrating infinite rigid classes through perturbations, and identifying one infinite nonrigid class: tetrahedra.

1. Introduction

It has been known since [7] and [3] that there are convex polyhedra, each of which may be unfolded to a planar polygon and then refolded to different convex polyhedra. For example, the cube may be unfolded to a "Latin cross" polygon, which may be refolded to 22 distinct non-cube convex polyhedra [5, Figs. 25.32-6]. But there has been only sporadic progress on understanding which pairs of convex polyhedra[1] have a common unfolding. A notable recent exception is the discovery [9] of a series of unfoldings of a cube that refold in the limit to a regular tetrahedron, partially answering Open Problem 25.6 in [5, p. 424].

Here we begin to explore a new question, which we hope will shed light on the unfold–refold spectrum of problems: Which polyhedra $\mathcal{P}$ are *refold-rigid* in the sense that any unfolding of $\mathcal{P}$ may only be refolded back to $\mathcal{P}$? The answer we provide here is: NONE—Every polyhedron $\mathcal{P}$ has an unfolding that refolds to an incongruent $\mathcal{P}'$. Thus every $\mathcal{P}$ may be *transformed* to some $\mathcal{P}'$.

This somewhat surprising answer leads to the next natural question: Suppose the unfoldings are restricted to *edge unfoldings*, those that only cut along edges of $\mathcal{P}$ (rather than permitting arbitrary cuts through the interior of faces). Say that a polyhedron $\mathcal{P}$ whose every edge unfolding only refolds back to $\mathcal{P}$ is *edge-refold rigid*, and otherwise is an *edge-refold transformer*. It was known that four of the five Platonic solids are edge-refold transformers (e.g., [4] and [8]). Here we prove

☆ This is the extended version of a EuroCG abstract (Demaine et al., 2012) [2].

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[1] All polyhedra in this paper are convex, and the modifier will henceforth be dropped.

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that the dodecahedron is edge-refold rigid: all of its edge unfoldings only fold back to the dodecahedron. The proof also demonstrates edge-refold rigidity for 11 of the Archimedean solids; we exhibit new refoldings of the truncated tetrahedron and the cuboctahedron. We also establish the same rigidity for infinite classes of slightly perturbed versions of these polyhedra. In contrast to this, we show that every tetrahedron is an edge-refold transformer: at least one among a tetrahedron's 16 edge unfoldings refolds to a different polyhedron.

This work raises many new questions, summarized in Section 6.

2. Notation and definitions

We will use $\mathcal{P}$ for a polyhedron in $\mathbb{R}^3$ and P for a planar polygon. An *unfolding* of a polyhedron $\mathcal{P}$ is a development of its surface after cutting to a single (possibly overlapping) polygon P in the plane. The surface of $\mathcal{P}$ must be cut open by a spanning tree of its vertices to achieve this. An *edge unfolding* only includes edges of $\mathcal{P}$ in its spanning cut tree. Note that we do not insist that unfoldings avoid overlap.

A *folding* of a polygon P is an identification of its boundary points that satisfies the three conditions of Alexandrov's theorem: (1) The identifications (or "gluings") close up the perimeter of P without gaps or overlaps; (2) The resulting surface is homeomorphic to a sphere; and (3) Identifications result in $\leqslant 2\pi$ surface angle glued at every point. Under these three conditions, Alexandrov's theorem guarantees that the folding produces a convex polyhedron, unique once the gluing is specified. See [1] or [5]. Note that there is no restriction that whole edges of P must be identified to whole edges, even when P is produced by an edge unfolding. We call a gluing that satisfies the above conditions an *Alexandrov gluing*.

A polyhedron $\mathcal{P}$ is *refold-rigid* if every unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$. Otherwise, $\mathcal{P}$ is a *transformer*. A polyhedron is *edge-refold rigid* if every edge unfolding of $\mathcal{P}$ may only refold back to $\mathcal{P}$, and otherwise it is an *edge-refold transformer*. Note we consider a polyhedron $\mathcal{P}$ a transformer if an unfolding can refold to an *incongruent* $\mathcal{P}'$. Some of our proofs establish a $\mathcal{P}'$ with more vertices than $\mathcal{P}$, so they are combinatorially different; some proofs establish the weaker incongruence.

3. Polyhedra are transformers

The proof that no polyhedron $\mathcal{P}$ is refold-rigid breaks naturally into two cases. We first state a lemma that provides the case partition. Let $\kappa(v)$ be the curvature at vertex $v \in \mathcal{P}$, i.e., the "angle gap" at v: 2π minus the total incident face angle $\alpha(v)$ at v. By the Gauss–Bonnet theorem, the sum of all vertex curvatures on $\mathcal{P}$ is 4π.

Lemma 1. *For every polyhedron $\mathcal{P}$, either there is a pair of vertices with $\kappa(a) + \kappa(b) > 2\pi$, or there are two vertices each with at most π curvature: $\kappa(a) \leqslant \pi$ and $\kappa(b) \leqslant \pi$.*

Proof. Suppose there is no pair with curvature sum more than 2π. So we have $\kappa(v_1)+\kappa(v_2) \leqslant 2\pi$ and $\kappa(v_3)+\kappa(v_4) \leqslant 2\pi$ for four distinct vertices. Suppose neither of these pairs have both vertices with at most π curvature. If $\kappa(v_2) > \pi$, then $\kappa(v_1) \leqslant \pi$; and similarly, if $\kappa(v_4) > \pi$, then $\kappa(v_3) \leqslant \pi$. Thus we have identified two vertices, v_1 and v_3, both with at most π curvature. □

We can extend this lemma to accommodate 3-vertex doubly covered triangles as polyhedra, because then every vertex has curvature greater than π.

Lemma 2. *Any polyhedron $\mathcal{P}$ with a pair of vertices with curvature sum more than 2π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*

Proof. Let $\kappa(a) + \kappa(b) > 2\pi$, and so the incident face angles satisfy $\alpha(a) + \alpha(b) < 2\pi$. Let γ be a shortest path on $\mathcal{P}$ connecting a to b. Cut open $\mathcal{P}$ with a cut tree T that includes γ as an edge. How T is completed beyond the endpoints of $\gamma = ab$ doesn't matter. (Recall our definition of unfolding does not demand non-overlap.)

Let γ_1 and γ_2 be the two sides of the cut γ, and let m_1 and m_2 be the midpoints of γ_1 and γ_2. Reglue the unfolding by folding γ_1 at m_1 and gluing the two halves of γ_1 together, and likewise fold γ_2 at m_2. All the remaining boundary of the unfolding outside of γ is reglued back exactly as it was cut by T. See Fig. 1.

The midpoint folds at m_1 and m_2 have angle π (because γ is a geodesic). The gluing draws the endpoints a and b together, forming a point with total angle $\alpha(a) + \alpha(b) < 2\pi$. Thus this gluing is an Alexandrov gluing, producing some polyhedron $\mathcal{P}'$. Generically $\mathcal{P}'$ has one more vertex than $\mathcal{P}$: it gains two vertices at m_1 and m_2, and a and b are merged to one. $\mathcal{P}'$ could only have the same number of vertices as $\mathcal{P}$ if $\alpha(a) + \alpha(b) = 2\pi$, which is excluded in this case. □

Lemma 3. *Any polyhedron $\mathcal{P}$ with a pair of vertices each with curvature at most π is not refold-rigid: There is an unfolding that may be refolded to a different polyhedron $\mathcal{P}'$.*