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Grid representations and the chromatic number ☆,☆☆

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ABSTRACT

A grid drawing of a graph maps vertices to the grid $\mathbb{Z}^d$ and edges to line segments that avoid grid points representing other vertices. We show that a graph G is q^d-colorable, d, $q \geqslant 2$, if and only if there is a grid drawing of G in $\mathbb{Z}^d$ in which no line segment intersects more than q grid points. This strengthens the result of D. Flores Peñaloza and F.J. Zaragoza Martinez. Second, we study grid drawings with a bounded number of columns, introducing some new NP-complete problems. Finally, we show that any planar graph has a planar grid drawing where every line segment contains exactly two grid points. This result proves conjectures asked by D. Flores Peñaloza and F.J. Zaragoza Martinez.

1. Introduction

Let $G = (V, E)$ be a simple, undirected and finite graph. A *k-coloring* of G is a function $f : V \to C$ for some set C of k colors such that $f(u) \neq f(v)$ for every edge $uv \in E$. If such a k-coloring of G exists, then G is *k-colorable*. The *chromatic number* $\chi(G)$ of G is the least k such that G is k-colorable. A complete graph on $n \in \mathbb{N}$ vertices is denoted as K_n.

Let $\overline{xy}$ denote the closed line segment joining two grid points $x, y \in \mathbb{Z}^d$. The line segment $\overline{xy}$ is *primitive* if $\overline{xy} \cap \mathbb{Z}^d = \{x, y\}$.

Definition 1. A grid drawing $\phi(G)$ of G in $\mathbb{Z}^d$ is an injective mapping $\phi : V \to \mathbb{Z}^d$ such that, for every edge $uv \in E$ and vertex $w \in V$, $\phi(w) \in \overline{\phi(u)\phi(v)}$ implies that $w = u$ or $w = v$.

That is, a grid drawing represents vertices of G by distinct grid points in $\mathbb{Z}^d$, and each edge by a line segment between its endpoints such that the only vertices an edge intersects are its own endpoints.

1.1. Robustness

In this paper we study the grid drawings from three points of view: robustness, compactness and planarity. The *robustness of a grid drawing* is understood as the maximum number of grid points a line segment of such a drawing intersects. For a given graph G we would like to find a grid drawing of G with the lowest robustness. In Section 2 we show a connection between such grid drawings and the chromatic number $\chi(G)$.

☆ A four-page abstract of an earlier version of this paper appeared in the proceedings of the 28th European Workshop on Computational Geometry EuroCG '12 (Balko, 2012) [2]. The booklet of abstracts is available here: http://www.diei.unipg.it/eurocg2012/booklet.pdf.

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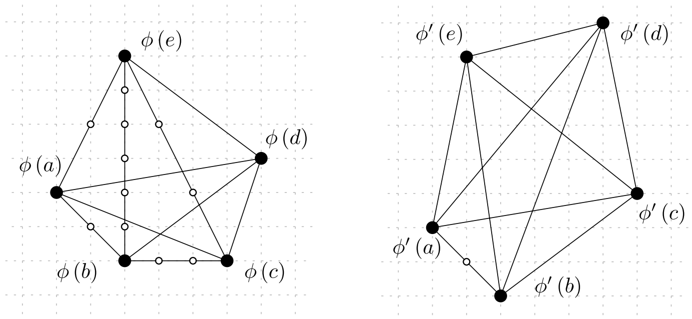

Fig. 1. Some grid drawings of the graph K_5 with the set of vertices $\{a, b, c, d, e\}$.

A graph G is said to be (*grid*) *locatable in* $\mathbb{Z}^d$ if there exists a grid drawing of G in $\mathbb{Z}^d$ where every edge is represented by a primitive line segment (such a drawing is also called *primitive*). Finding a primitive grid drawing of G is called *locating the graph* G. D. Flores Peñaloza and F.J. Zaragoza Martinez [7] showed the following characterization:

Theorem 1. *(See [7].) A graph G is locatable in $\mathbb{Z}^2$ if and only if G is 4-colorable.*

Therefore not all graphs are locatable in $\mathbb{Z}^2$ and every (two-dimensional) grid drawing of any k-colorable graph, where $k > 4$, contains a line segment which intersects at least three grid points. This leads us to a generalization of the concept of locatability. Let the number $gp(\phi(G))$ denote the maximum number of grid points any line segment of the grid drawing $\phi(G)$ intersects.

Definition 2. A graph G is (*grid*) *q-locatable* in $\mathbb{Z}^d$, for some integer $q \geqslant 2$, if there exists a grid drawing $\phi(G)$ in $\mathbb{Z}^d$ such that $gp(\phi(G)) \leqslant q$.

The *grid robustness of a graph* G is the minimum of $gp(\phi(G))$ among all grid drawings $\phi(G)$. For example, the graph K_5 has chromatic number five, thus it is not (two-)locatable in $\mathbb{Z}^2$. However the second grid drawing in Fig. 1 shows that K_5 is three-locatable in $\mathbb{Z}^2$ (the third grid point on a line segment is denoted by an empty circle). The main result of Section 2 is a stronger version of Theorem 1.

Theorem 2. *For integers $d, q \geqslant 2$, a graph G is q^d-colorable if and only if G is q-locatable in $\mathbb{Z}^d$.*

The proof is constructive and for a given coloring of G with at most q^d colors we get a linear time algorithm which gives us a grid drawing $\phi(G)$ of G in $\mathbb{Z}^d$ where every line segment contains at most q grid points. Conversely, we also show that if we have such a grid drawing of G, then it is quite trivial to get a coloring of G with at most q^d colors.

Corollary 1. *For an integer $d \geqslant 2$, a graph is 2^d-colorable if and only if it is locatable in $\mathbb{Z}^d$.*

1.2. Compactness

The compactness of a grid drawing signifies bounded space of the drawing in some fashion. Our aim in Section 3 is to draw (or locate) the graph in the smallest number of columns. For an integer $d \geqslant 2$, a *column* in the grid $\mathbb{Z}^d$ with the *rank* $(x_1, \ldots, x_{d-1}) \in \mathbb{Z}^{d-1}$ is the set $\{(x_1, \ldots, x_{d-1}, x) \mid x \in \mathbb{Z}\}$. It turns out that this problem is closely related to a special form of a defective coloring in which some colors induce independent sets and some disjoint unions of paths (linear forests). We call these colorings *mixed*.

Definition 3. If there is a mixed coloring of a graph G such that a colors induce independent sets (so-called *normal colors*) and b colors induce linear forests (*path colors*), then we say that G is (a, b)-*colorable*.

In the case of primitive grid drawings we obtain the following characterization:

Theorem 3. *For a graph G, integers $d \geqslant 2$ and l, $2^{d-1} < l \leqslant 2^d$, the following statements are equivalent*:

1. *G can be located on l columns in $\mathbb{Z}^d$,*
2. *G is $(2l - 2^d, 2^d - l)$-colorable.*