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# Bayes estimation for the Marshall–Olkin bivariate Weibull distribution

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The paper is dedicated to Professor C.R. Rao on his 90th birthday.

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ABSTRACT

In this paper, we consider the Bayesian analysis of the Marshall–Olkin bivariate Weibull distribution. It is a singular distribution whose marginals are Weibull distributions. This is a generalization of the Marshall–Olkin bivariate exponential distribution. It is well known that the maximum likelihood estimators of the unknown parameters do not always exist. The Bayes estimators are obtained with respect to the squared error loss function and the prior distributions allow for prior dependence among the components of the parameter vector. If the shape parameter is known, the Bayes estimators of the unknown parameters can be obtained in explicit forms under the assumptions of independent priors. If the shape parameter is unknown, the Bayes estimators cannot be obtained in explicit forms. We propose to use the importance sampling method to compute the Bayes estimators and also to construct associated credible intervals of the unknown parameters. The analysis of one data set is performed for illustrative purposes. Finally we indicate the analysis of data sets obtained from series and parallel systems.



## 1. Introduction

Exponential distribution has been used extensively for analyzing the univariate lifetime data, mainly due to its analytical tractability. A huge amount of work on exponential distribution has been found in the statistical literature. Several books and book chapters have been written exclusively on exponential distribution, see for example Balakrishnan and Basu (1995), Johnson et al. (1995) etc. A variety of bivariate (multivariate) extensions of the exponential distribution have also been considered in the literature. These include the distributions of Gumbel (1960), Freund (1961), Henrich and Jensen (1995), Marshall and Olkin (1967) and Downton (1970) as well as Block and Basu (1974), see for example Kotz et al. (2000).

Among several bivariate (multivariate) exponential distributions, Marshall–Olkin bivariate exponential (MOBE), see Marshall and Olkin (1967), has received the maximum attention. The MOBE distribution is the only bivariate exponential distribution with exponential marginals and it also has the bivariate lack of memory property. It has a nice physical interpretation based on random shocks. Extensive work has been done in developing the inference procedure of the MOBE model and its characterization. Kotz et al. (2000) provided an excellent review on this distribution till that time, see also Karlis (2003) and Kundu and Dey (2009) in this respect.

The MOBE distribution is a singular distribution, and due to this reason, it has been used quite successfully when there are ties in the data set. The marginals of the MOBE distribution are exponential distributions, and definitely that is one of the major limitations of the MOBE distribution. Since the marginals are exponential distributions, if it is known (observed) that the estimated probability density functions (PDFs) of the marginals are not decreasing functions or the hazard functions are not constant, then MOBE cannot be used. Marshall and Olkin (1967) proposed a more flexible bivariate model, namely

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the Marshall–Olkin bivariate Weibull (MOBW) model, where the marginals are Weibull models. Therefore, if it is observed empirically that the marginals are decreasing or unimodal PDFs and monotone hazard functions, then MOBW models can be used quite successfully. Further, MOBW can also be given a shock model interpretation.

Although, extensive work has been done on the MOBE model, the MOBW model has not received much attention primarily due to the analytical intractability of the model, see for example Kotz et al. (2000). Lu (1992) considered the MOBW model and proposed the Bayes estimators of the unknown parameters. Recently, Kundu and Dey (2009) proposed an efficient estimation procedure to compute the maximum likelihood estimators (MLEs) using the expectation maximization (EM) algorithm, which extends the EM algorithm proposed by Karlis (2003) to find the MLEs of the MOBE model. Although the EM algorithm proposed by Kundu and Dey (2009) works very well even for small sample sizes, it is well known that the MLEs do not always exist. Therefore, in those cases the method cannot be used.

The main aim of this paper is to develop Bayesian inference for the MOBW model. We want to compute the Bayes estimators and the associated credible intervals under proper priors. The paper is closely related to the paper by Pena and Gupta (1990). Pena and Gupta (1990) obtained the Bayes estimators of the unknown parameters of the MOBE model for both series and parallel systems under a quadratic loss function. They have used a very flexible Dirichlet-Gamma conjugate prior. Depending on the hyper-parameters, the Dirichlet-Gamma prior allows the stochastic dependence and independence among model parameters. Moreover, Jeffrey's non-informative prior can also be obtained as a limiting case. The Bayes estimators can be obtained in explicit forms, and Pena and Gupta (1990) provided a numerical procedure to construct the highest posterior density (HPD) credible intervals of the unknown parameters.

The MOBW model proposed by Marshall and Olkin (1967) has a common shape parameter. If the common shape parameter is known, the same Dirichlet-Gamma prior proposed by Pena and Gupta (1990) can be used as a conjugate prior, but if the common shape parameter is not known, then as expected the conjugate priors do not exist. We propose to use the same conjugate prior for the scale parameters, even when the common shape parameter is unknown. We do not use any specific form of prior on the shape parameter. It is only assumed that the PDF of the prior distribution is log-concave on $(0, \infty)$. It may be assumed that the assumption of log-concave PDF of the prior distribution is not very uncommon, see for example Berger and Sun (1993), Mukhopadhyay and Basu (1997), Patra and Dey (1999) or Kundu (2008). Moreover, many common distribution functions, for example normal, log-normal, gamma and Weibull distributions have log-concave PDFs.

Based on the above prior distribution we obtain the joint posterior distribution of the unknown parameters. As expected the Bayes estimators cannot be obtained in closed form. We propose to use the importance sampling procedure to generate samples from the posterior distribution function and in turn use them to compute the Bayes estimators and also to construct the posterior density credible intervals of the unknown parameters. It is observed that the Bayes estimators exist even when the MLEs do not exist. We compare the Bayes estimators and the credible intervals with the MLEs and the corresponding confidence intervals obtained using the asymptotic distribution of the MLEs, when they exist. It is observed that when we have non-informative priors, then the performances of the Bayes estimators and the MLEs are quite comparable, but with informative priors, the Bayes estimators perform better than the MLEs as expected. For illustrative purposes, we have analyzed one data set.

Then we consider the Bayes estimators of the MOBW parameters, when random samples from series systems are available. If the two components are connected in series, and their lifetimes are denoted by $X_1$ and $X_2$ respectively, then the random vector observed on system failure is $(Z, \Delta)$, where

$$Z = \min\{X_1, X_2\}, \qquad \Delta = \begin{cases} 0 & \text{if } X_1 = X_2 \\ 1 & \text{if } X_1 < X_2 \\ 2 & \text{if } X_1 > X_2. \end{cases} \tag{1}$$

Based on the assumption that $(X_1, X_2)$ has a MOBW distribution, we develop the Bayesian inference of the unknown parameters, when we observe the data of $(Z, \Delta)$ as described above. It is observed that the Bayes estimators cannot be obtained in explicit forms, and we provide an importance sampling procedure to compute the Bayes estimators and also to construct the associated HPD credible intervals.

We further consider the Bayes estimators of the MOBW parameters, when the data are obtained from a parallel system. If the system consists of two components and they are connected in parallel, and if it is assumed that the lifetimes of the two components are $X_1$ and $X_2$, then the random vector observed on system failure is $(W, \Delta)$, where

$$W = \max\{X_1, X_2\}, \qquad \Delta = \begin{cases} 0 & \text{if } X_1 = X_2 \\ 1 & \text{if } X_1 < X_2 \\ 2 & \text{if } X_1 > X_2. \end{cases} \tag{2}$$

In this case also, based on the assumption that $(X_1, X_2)$ has a MOBW distribution, we develop the Bayesian inference of the unknown parameters based on importance sampling.

The rest of the paper is organized as follows. In Section 2, we briefly describe the MOBW model. The necessary prior assumptions are presented in Section 3. Posterior analysis and Bayesian inference are presented in Section 4. The analysis of a data set is presented in Section 5. In Section 6, we consider the Bayesian inference of the unknown parameters, when we observe the data from a series or from a parallel system. Finally we conclude the paper in Section 7.