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On Crevecoeur's bathtub-shaped failure rate model

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ABSTRACT

Crevecoeur (1993) developed a three-parameter bathtub-shaped failure rate model that enjoys nice mathematical properties and justification from engineering perspectives. In this paper, we derive the explicit formulas for the maximum likelihood estimation (MLE) of parameters for his model applied to both non-censored data and right-censored data. Meanwhile, explicit formulas for the MLE of parameters of Xie–Tang–Goh's model (Xie et al., 2002) are given for both types of data in the paper. The results from using these two models are compared to some real data sets both in terms of AIC values and in terms of how well the intensity is fitted. We also investigate the MLE-based statistical inference including parameter confidence intervals and parameter significance test for both models. Finally, aiming at reliability-related decision-making and predicting the evolution behavior of a system, we report the relations of the reliability characteristics during the improvement phase to those of the steady service phase. A theory of system improvement limit is presented based on Crevecoeur's failure rate model.

1. Introduction

Many lifetime failure data have a bathtub-shaped *failure rate*. Recall for a repairable system, failure rate means the failure intensity function, or the rate of occurrence of failure (ROCOF); while for a non-repairable system, failure rate means the hazard rate function. Various mathematical models of bathtub-shaped failure rate are studied in a large amount of literature. Many of those models are Weibull-distribution-related or Weibull-process related models, such as those in Lee (1980), Hjorth (1980), Mudholkar and Srivastava (1993), Crevecoeur (1993), Jiang and Murthy (1995b), Xie and Lai (1996), Xie et al. (2002) and Carrasco et al. (2008). A four-parameter model is studied in Wang (2000), and a two-parameter model can be found in Chen (2000). A new inverted U-shaped model was proposed in Avinadav and Raz (2008) which is different from the common inverted U-shaped model such as the log-normal and the log-logistic models (Lawless, 1984). Some new models for lifetime data have been proposed very recently such as in Silva et al. (2010), Paranaíba et al. (2011), Hemmati et al. (2011) and Cancho et al. (2011). A nice survey on bathtub shaped failure rate models can be found in Lai et al. (2001). An earlier review can be found in Rajarshi and Rajarshi (1988).

In this paper, we shall exclusively focus on Crevecoeur's model (Crevecoeur, 1993), and at the same time, compare Crevecoeur's model with Xie–Tang–Goh's model (Xie et al., 2002). This is because we are able to find the explicit formula for the MLE of parameters for both models applied to both non-censored data and right-censored data.

Crevecoeur (1993) initially considered the positive and negative feedback mechanisms of a repairable system from the engineering point of view and consequently obtained the following bathtub-shaped failure rate function $\lambda(t)$ defined at

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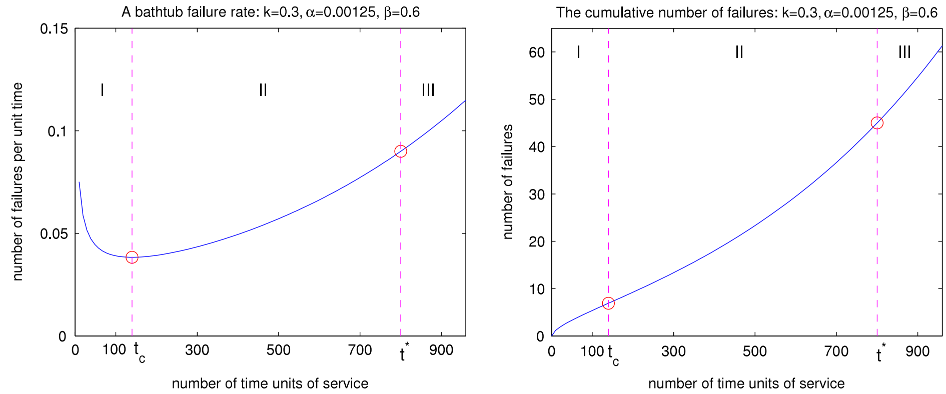

Fig. 1.1. Left: Crevecoeur's failure rate $\lambda(t)$. Right: Crevecoeur's mean cumulative number of failures $\Lambda(t)$.

$t \geq 0$ by

$$\lambda(t) = k\beta t^{\beta-1} \mathrm{e}^{\alpha t}(1 + \alpha t/\beta), \quad k > 0,\ 0 < \alpha \ll \beta < 1, \tag{1.1}$$

where, k, α, β are parameters of the system, and t is the running time of the system.

The left panel of Fig. 1.1 is an example of the failure rate function with $k = 0.03$, $\alpha = 0.00125$, and $\beta = 0.6$. Recall that if $N(t)$ denotes the random variable—*cumulative number of failures* till t then the *mean cumulative number of failures* till t is defined as

$$\Lambda(t) := \mathcal{E}(N(t)) = \int_0^t \lambda(s)\,\mathrm{d}s. \tag{1.2}$$

Specifically, for Crevecoeur's model, we have that

$$\Lambda(t) = kt^{\beta} \mathrm{e}^{\alpha t}. \tag{1.3}$$

The graph of $\Lambda(t)$ is shown on the right panel of Fig. 1.1.

Parameter estimation is always of interest for a given model. In this regard, two parameter estimation methods are mainly employed: the graphical method and the *maximum likelihood estimation*. Crevecoeur did not discuss the maximum likelihood estimation of his model (Eq. (1.1)). No closed or explicit formulas exist for obtaining the MLE under the circumstances studied in Xie et al. (2002) and Lai et al. (2003). On the other hand, in the papers cited aforementioned and in Jiang and Murthy (1995a), Jiang and Murthy (1999) and Jiang and Kececioglu (1992), using a graphical method was focused. *Unlike those papers, this paper derives explicit formulas to obtain the MLE for Crevecoeur's and Xie–Tang–Goh's models applied to both non-censored and right-censored data*. Those results are stated in Theorems 2.1 and 2.3. Estimating parameter values with explicit formulas may avoid some disadvantages suffered by approaches requiring iterations. Some optimization algorithms may be sensitive to the specified initial parameter values. It may not be reliable to pre-specify a parameter domain region (either bounded or not) to search for the parameter value that will produce the extreme objective function value, especially when there are more than one parameter with the entire real domain. Moreover, numerical errors may mistakenly disqualify the promising combination of parameter values and/or qualify the worse combination due to the possible sensitivity of some parameter to a given model. The closed MLE formulas for Crevecoeur's model applied to non-censored data was initially given by one of the authors (Wang, 1997). In a recent study, the closed formulas for the MLE of the parameters in both the non-censored and right-censored Weibull processes with incomplete early observations are given in Yu et al. (2008). When the MLE of parameters must be derived numerically, the expectation–maximization (EM) method has become a popular tool, especially when involving incomplete data; for example, see Nandi and Dewan (2010) and the references cited therein.

In Fig. 1.1, the time instant

$$t^* := 1/\alpha \tag{1.4}$$

is regarded as the *instability instant* of the system, or, the *lifetime* of the system. This is due to the widely accepted fact that, a natural phenomenon described by an exponential function is unstable when the exponent is greater than 1. For an aging repairable system, t^* can be regarded as the *overhaul* or *discarding time*. In Crevecoeur (1993) Crevecoeur compared the experimental results of the *necking* time of a creeping metal to the results computed by Eq. (1.4) and concluded that the necking of the metal occurs at about the instability instant t^*. His study of using Eq. (1.3) on the failure data of the main propulsion engines of US army recorded by Ascher and Feingold (1984, p. 75) also confirmed that the value of t^* computed by Eq. (1.4) coincides with the recorded data.

The other quantity t_c in Fig. 1.1 is the critical point of the curve of $\lambda(t)$. It is the time instant where the reliability of the system transits from the improvement phase to the steady service phase. We shall call t_c *the transit instant* of the system. It