Discrete Applied Mathematics 159 (2011) 246–250

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

Hamiltonian index is NP-complete

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ARTICLE INFO

Article history:
Received 23 June 2009
Received in revised form 19 August 2010
Accepted 31 August 2010
Available online 29 September 2010

Keywords:
Complexity
Hamiltonian index
Line graph
Branch-bond
NP-complete problem

ABSTRACT

In this paper we show that the problem to decide whether the hamiltonian index of a given graph is less than or equal to a given constant is NP-complete (although this was conjectured to be polynomial). Consequently, the corresponding problem to determine the hamiltonian index of a given graph is NP-hard. Finally, we show that some known upper and lower bounds on the hamiltonian index can be computed in polynomial time.

1. Introduction

By a *graph* we mean a simple loopless finite undirected graph $G = (V(G), E(G))$. If we admit G to have multiple edges, we say that G is a *multigraph*. A graph is *trivial* if it has only one vertex. For a graph G and a nonnegative integer k, we denote $V_k(G) = \{x \in V(G) \mid d_G(x) = k\}$, where $d_G(x)$ is the degree of x in G. A graph G is *d-regular* if $V(G) = V_d(G)$, and in the special case $d = 3$ we say that G is *cubic*. The *maximum degree* (*minimum degree*) of a graph G is denoted by $\Delta(G)$ ($\delta(G)$), and G is said to be *even* if every vertex of G has even degree.

A spanning subgraph of a graph G is also referred to as a *factor* of G. A subgraph of G is called *eulerian* if it is connected and even, and G is *supereulerian* if G has an eulerian factor. A 2-*factor* is a factor with all vertices of degree 2. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. When we simply say F is an *induced subgraph* of G, it means that F is induced by its set of vertices.

If $F \subset G$ is a connected subgraph of G, we say that a graph H is obtained from G by *contracting* the subgraph F, if $V(H) = (V(G) \setminus V(F)) \cup \{v_F\}$ (where $v_F \notin V(G)$) and $E(H) = (E(G) \setminus E(F)) \cup \{v_F u : u \in V(G) \setminus V(F)$ and $xu \in E(G)$ for some $x \in V(F)\}$.

The *line graph* $L(G)$ of a graph $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges have a common vertex in G. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$. Chartrand [2] showed that if a connected graph G is not a path, then $L^m(G)$ is hamiltonian (and hence also has a 2-factor, an even factor and an eulerian factor) for some integer m. The *hamiltonian index*

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doi:10.1016/j.dam.2010.08.027

(2-*factor index*, *even factor index*, *supereulerian index*) of a graph G, denoted by $h(G)$ ($f(G)$, $ef(G)$, $s(G)$), is the smallest integer m such that $L^m(G)$ contains a hamiltonian cycle (2-factor, even factor, spanning eulerian subgraph), respectively.

For further graph-theoretical notations and terminology not defined here we refer the reader to [7], and for complexity concepts we refer to [3].

In [11], it was conjectured that there is an algorithm to determine the hamiltonian index of a graph G with $h(G) \geq 2$ in polynomial time. In Section 2, we will prove that the problem to determine the hamiltonian index $h(G)$ of a graph G is NP-hard even for graph G with large $h(G)$, and we state two NP-complete formulations of the problem (which disproves the conjecture if P $\neq$ NP). In Section 3 we show that the upper and lower bounds for $h(G)$, $f(G)$, $ef(G)$ and $s(G)$, given in [8,10,9,12], can be determined in polynomial time.

2. NP-completeness of the hamiltonian index

Let G be a graph. For any two subgraphs H_1 and H_2 of G, define the *distance* $d_G(H_1, H_2)$ between H_1 and H_2 as the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. If $d_G(e, H) = 0$ for an edge e of G, we say that *H dominates e*. A subgraph H of G is called *dominating* if it dominates all edges of G. There is a characterization of graphs G with $h(G) \leq 1$ which involves the existence of a dominating eulerian subgraph in G.

Theorem 1 (*Harary and Nash-Williams, [5]*). *Let G be a graph with at least three edges. Then $h(G) \leq 1$ if and only if G has a dominating eulerian subgraph.*

A *branch* in G is a nontrivial path in G with endvertices in $V(G) \setminus V_2(G)$ and with interior vertices (if any) in $V_2(G)$. We use $\mathcal{B}(G)$ to denote the set of branches of G and $\mathcal{B}_1(G)$ to denote the subset of $\mathcal{B}(G)$ in which at least one endvertex has degree one. For a subgraph H of G, $\mathcal{B}_H(G)$ denotes the set of branches of G whose edges are all in H.

The following theorem can be considered as an analogue of Theorem 1 for the k-iterated line graph $L^k(G)$ of a graph G.

Theorem 2 (*Xiong and Liu, [11]*). *Let G be a connected graph that is not a path and let $k \geq 2$ be an integer. Then $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$ where $EU_k(G)$ denotes the set of those subgraphs H of G which satisfy the following five conditions:*

(I) *H is an even graph;*
(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;
(III) $d_G(H_1, H - H_1) \leq k - 1$ *for every induced subgraph H_1 of H with* $\emptyset \neq V(H_1) \subsetneq V(H)$;
(IV) $|E(B)| \leq k + 1$ *for every branch* $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$;
(V) $|E(B)| \leq k$ *for every branch B in* $\mathcal{B}_1(G)$.

For a graph G and a positive integer $m \geq 2$, let $\Gamma_1^{(m)}(G), \Gamma_2^{(m)}(G), \ldots, \Gamma_s^{(m)}(G)$ denote the components of the graph obtained from G by removing edges and interior vertices of all branches of length at least m (some $\Gamma_i^{(m)}(G)$ can be trivial), and $H^{(m)}(G)$ the graph obtained from G by contracting the subgraphs $\Gamma_1^{(m)}(G), \Gamma_2^{(m)}(G), \ldots, \Gamma_s^{(m)}(G)$ to distinct vertices. The vertex of $H^{(m)}(G)$ obtained by contracting a subgraph $\Gamma_i^{(m)}(G)$ of G will be denoted by Γ_i. (Note that $\Gamma_i^{(m)}(G)$ are induced subgraphs of G, and $H^{(m)}(G)$ is a graph, i.e., has no parallel edges).

Now we construct a (multi)graph $\tilde{H}^{(m)}(G)$ from $H^{(m)}(G)$ by the following construction:

(1) Delete all cycles with all vertices except one of degree 2 (i.e., all cycles that are endblocks).
(2) If $\Gamma_i, \Gamma_j \in V(H^{(m)}(G))$ are connected by n_1 branches of length m or $m + 1$ and m_1 branches of length at least $m + 2$ with $m_1 + n_1 \geq 3$, then delete some of them in such a way that there remain n_2 branches of length m or $m + 1$ and m_2 branches of length at least $m + 2$, where

$$(m_2, n_2) = \begin{cases} (2, 0) & m_1 \ even,\ n_1 = 0; \\ (1, 0) & m_1 \ odd,\ n_1 = 0; \\ (1, 1) & n_1 = 1; \\ (0, 2) & n_1 \geq 2. \end{cases}$$

(3) Delete all end-branches of length m.
(4) Replace each non-end branch of length m or $m + 1$ by a single edge.

Theorem 3 (*Hong et al. [6]*). *If G is a graph with $\Delta(G) \geq 3$ and $h(G) \geq 2$, then*

$$h(G) = \min\{m \geq 2 : \tilde{H}^{(m)}(G) \text{ has a spanning eulerian subgraph}\}.$$

The Hamiltonian Problem (HP), i.e., the problem to decide whether a given graph is hamiltonian, is one of the classical NP-complete problems. The following two results show that the HP remains NP-complete even if restricted to cubic graphs (the Cubic Hamiltonian Problem, CHP), or to line graphs (the Line Graph Hamiltonian Problem, LHP).

Theorem 4 (*Garey et al. [4]*). *It is NP-complete to decide whether a given cubic graph is hamiltonian.*