Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Applied Mathematics 148 (2005) 89–105

DISCRETE APPLIED MATHEMATICS

www.elsevier.com/locate/dam

On monochromatic component size for improper colourings

Keith Edwards[a], Graham Farr[b,1]

[a]*Division of Applied Computing, University of Dundee, Dundee DD1 4HN, UK*
[b]*School of Computer Science and Software Engineering, Monash University (Clayton Campus), Wellington Road, Clayton, Victoria 3800, Australia*

Received 17 December 2002; received in revised form 7 October 2003; accepted 12 October 2004

Abstract

This paper concerns improper λ-colourings of graphs and focuses on the sizes of the monochromatic components (i.e., components of the subgraphs induced by the colour classes). Consider the following three simple operations, which should, heuristically, help reduce monochromatic component size: (a) assign to a vertex the colour that is least popular among its neighbours; (b) change the colours of any two adjacent differently coloured vertices, if doing so reduces the number of monochromatic edges; and (c) change the colour of a vertex, if by so doing you can reduce the size of the largest monochromatic component containing it without increasing the number of monochromatic edges. If a colouring cannot be further improved by these operations, then we regard it as locally optimal. We show that, for such a locally optimal 2-colouring of a graph of maximum degree 4, the maximum monochromatic component size is $O(2^{(2\log_2 n)^{1/2}})$. The operation set (a)–(c) appears to be one of the simplest that achieves a $o(n)$ bound on monochromatic component size. Recent work by Alon, Ding, Oporowski and Vertigan, and then Haxell, Szabó and Tardos, has shown that some algorithms can do much better, achieving a constant bound on monochromatic component size. However, the simplicity

[1] Some of the work of this paper was done while Farr was visiting The Department of Applied Computing, University of Dundee; Institut für Computergraphik und Algorithmen, Technische Universität Wien; and Department of Computer Science, Royal Holloway, University of London.

E-mail address: gfarr@csse.monash.edu.au (G. Farr).

doi:10.1016/j.dam.2004.10.005

of our operation set, and of the associated local search algorithm, make the algorithm, and our locally optimal colourings, of interest in their own right.

Keywords: Graph colouring; Graph partition; Monochromatic components; Metacolouring; Algorithm

1. Introduction

An *assignment*, or λ-*assignment*, of a graph G is a map from $V(G)$ to some set Λ of colours such as $\{1, \ldots, \lambda\}$; i.e., it is a 'colouring' that may be improper. We will use terminology from graph colouring for such maps where the meaning is clear. For example, a *colour class* is a preimage, under the λ-assignment, of a single colour. The colour class for colour $i \in \Lambda$ is denoted by C_i.

A *chromon* of G under an assignment c is a component of a subgraph of G induced by a colour class, or in other words, a maximal connected monochromatic subgraph (sometimes called a *monochromatic component*). A *k-chromon* is a chromon with k vertices.

This paper mainly concerns 2-assignments, for a graph, which are locally optimal (in a natural sense) with respect to maximum chromon size.

A graph G is $[\lambda, C]$-*colourable* if it has a λ-assignment in which every chromon has at most C vertices. An ordinary (proper) colouring is thus a $[\lambda, 1]$-colouring, and the chromons under such a colouring are just the individual vertices.

A class of graphs Γ is $[\lambda, C]$-*colourable* if every $G \in \Gamma$ is $[\lambda, C]$-colourable. Γ is λ-*metacolourable* if there exists C such that Γ is $[\lambda, C]$-colourable. The *metachromatic number* $\boldsymbol{\chi}(\Gamma)$ is the smallest λ such that Γ is λ-metacolourable.

This is similar in spirit to the concept of fragmentability of classes of graphs that we introduced in [9]. In fact, it is to fragmentability as ordinary graph colouring is to independent sets. This paper, though, does not depend on that one.

Γ_d denotes the class of graphs of maximum degree $\leqslant d$.

In our main result, we give a simple local search algorithm for finding a 2-assignment of a graph of maximum degree 4 in which all chromons have size $O(2^{(2\log_2 n)^{1/2}})$. The algorithm uses three simple operations, involving changing the colour of just one or two vertices at a time, and appears to be one of the simplest algorithms that attain maximum chromon size $o(n)$. Our bound on chromon size applies to any 2-assignment that cannot be improved by applying any of our three operations. Since these operations are arguably the most natural local operations that can be done in this situation, these 2-assignments are worth studying. We also note that some such 2-assignments have maximum chromon size within a constant factor of our upper bound.

Alon et al. [4] show that (in our notation) Γ_4 is $[2, 57]$-colourable. They do not present an algorithm explicitly, though it is reasonable to expect that their approach would yield one. More recently, Haxell et al. [12, Section 2.2] have shown that Γ_4 is $[2, 6]$-colourable, and their proof yields an efficient algorithm. Alon (private communication, 2002) also reports that an algorithm with some constant bound on chromon size can be obtained using ideas in [3].