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Discrete Applied Mathematics 148 (2005) 240–245
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

A solvable case of image reconstruction in discrete tomography

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Received 3 October 2003; received in revised form 24 February 2005; accepted 17 March 2005
Available online 27 April 2005

Abstract

A graph-theoretical model is used to show that a special case of image reconstruction problem (with 3 colors) can be solved in polynomial time.

For the general case with 3 colors, the complexity status is open. Here we consider that among the three colors there is one for which the total number of multiple occurrences in a same line (row or column) is bounded by a fixed parameter. There is no assumption on the two remaining colors.

Keywords: Complete bipartite graph; Edge coloring; Perfect matchings; Discrete tomography

1. Introduction

We shall consider a special case of image reconstruction problem in discrete tomography.

The formulation will be based on graph-theoretic concepts (see [2]) and this will allow us to show that this case can be solved in polynomial time; it generalizes earlier known cases where the problem can be solved. The complexity status of a slight extension of this solvable case is still open; so our result is a step towards the boundary between easy and

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doi:10.1016/j.dam.2005.03.006

difficult problems in image reconstruction. The reader is referred to [5] for definitions about complexity.

We shall now define the general image reconstruction problem as follows: an image of $(m \times n)$ pixels of p different colors has to be reconstructed. For convenience we consider that there is in addition a color $p+1$ which is the ground color. We are given the number $\alpha(i,s)$ of pixels of each color s in each row i and also the number $\beta(j,s)$ of pixels of each color s in each column j; is it possible to reconstruct an image, i.e., can one assign a color s to each entry (i,j) of the image in such a way that there are $\alpha(i,s)$ occurrences of color s in each row i and $\beta(j,s)$ occurrences of color s in each column j, for all i, j, s?

This simplified version of image reconstruction problems occurring in discrete tomography is denoted by $R(m,n,p)$; it is a combinatorial problem whose complexity status is unknown for $p=2$ colors (i.e., when we have $p+1=3$ colors including the ground color). It is NP-complete for $p \geqslant 3$ (see [3,6]). In [4] some special cases solvable in polynomial time have been presented. Notice that it is solvable in polynomial time if $p+1=2$ (see [7]).

For a solution to exist we must necessarily have

$$\sum_{s=1}^{p+1} \alpha(i,s) = n, \quad i = 1, \ldots, m,$$

$$\sum_{s=1}^{p+1} \beta(j,s) = m, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} \alpha(i,s) = \sum_{j=1}^{n} \beta(j,s), \quad s = 1, \ldots, p+1.$$

These conditions are necessary but not sufficient for the existence of a solution to $R(m,n,p)$.

2. Graph-theoretical formulation

We associate with the problem a complete bipartite graph $G = K_{m,n}$ on two sets of nodes R, S with sizes m and n. Each edge $[i,j]$ of $K(m,n)$ corresponds to entry (i,j) in row i and column j of the $(m \times n)$ array.

The image reconstruction problem can be interpreted as follows: The entries of color s in the array correspond to a subset B_s of edges (a partial subgraph of $K(m,n)$ such that B_s has $\alpha(i,s)$ edges adjacent to node i of R and $\beta(j,s)$ edges adjacent to node j of S. We have to find a partition $B_1, B_2, \ldots, B_{p+1}$ of the edge set of $K(m,n)$ where each B_s satisfies the above degree requirements.

As mentioned above, for the case $p+1=3$, the complexity is unknown. The problem is solvable in polynomial time with $p+1=4$ colors (see [4]) if three colors, say colors 1, 2 and 3, are *unary*, i.e. $\alpha(i,s) \leqslant 1$, $\beta(j,s) \leqslant 1$ for $s = 1, 2, 3$, and all i, j. In the same paper, it is shown that it is solvable with $p+1=3$ colors if two colors, say colors 1 and 2, are *semi-unary*, i.e. $\alpha(i,1) \leqslant 1\ \forall i$ or $\beta(j,1) \leqslant 1\ \forall j$, and $\alpha(i,2) \leqslant 1\ \forall i$ or $\beta(j,2) \leqslant 1\ \forall j$.