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Data-independent neighborhood functions and strict local optima

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Abstract

The paper proves that data-independent neighborhood functions with the smooth property (all strict local optima are global optima) for maximum 3-satisfiability (MAX 3-SAT) must contain all possible solutions for large instances. Data-independent neighborhood functions with the smooth property for 0–1 knapsack are shown to have size with the same order of magnitude as the cardinality of the solution space. Data-independent neighborhood functions with the smooth property for traveling salesman problem (TSP) are shown to have exponential size. These results also hold for certain polynomially solvable sub-problems of MAX 3-SAT, 0–1 knapsack and TSP.

Keywords: Computational complexity; Local search algorithm; Exponential neighborhood

1. Introduction

The effectiveness of local search algorithms [8] on discrete optimization problems is highly dependent on the choice of neighborhood function. This paper proves that the only data-independent neighborhood functions with the smooth property (all strict local optima are global optima) for maximum 3-satisfiability (MAX 3-SAT) [4] are neighborhood functions that contain all possible solutions for large instances. More precisely, if a given neighborhood function for MAX 3-SAT has the smooth property, then, for instances with

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$n \geqslant 4$ Boolean variables, the neighborhood of every solution $\boldsymbol{x}$ contains all possible solutions except for the solution $\boldsymbol{x}$ itself. A result for 0–1 knapsack shows that data-independent neighborhood functions with the smooth property must have size that is $\Theta(2^n)$, where n denotes the number of items in the problem instance. Furthermore, a neighborhood function η^{K} with the smooth property for 0–1 knapsack is given so that if $\eta(I, \boldsymbol{x}) \subset \eta^{\mathrm{K}}(I, \boldsymbol{x})$ for some instance I and solution $\boldsymbol{x}$, then η does not have the smooth property. The neighborhood function η^{K} is said to be the minimal data-independent neighborhood function with the smooth property for 0–1 knapsack. Note that a neighborhood function η^{MS} consisting of all solutions for instances of MAX 3-SAT (with $n \geqslant 4$ Boolean variables) also has the property that if $\eta(I, \boldsymbol{x}) \subset \eta^{\mathrm{MS}}(I, \boldsymbol{x})$ (for an instance I with $n \geqslant 4$ Boolean variables) and solution $\boldsymbol{x}$, then η does not have the smooth property. This paper also shows that if a given neighborhood function of traveling salesman problem (TSP) [6] has the smooth property, then the neighborhood of every solution has cardinality $\Omega(2^{n/3})$, where n denotes the number of cities in the problem instance.

The results in this paper are obtained by constructing instances of the discrete optimization problem such that specified data-independent neighborhood functions have a strict local optimum that is not a global optimum. In particular, instances are created where there is a unique global optimum and a unique solution with the second best objective function value. The solution with the second best objective function value is chosen such that the unique global optimum is not in its neighborhood. This implies that the solution with the second best objective function value is a strict local optimum. By construction, the classes of instances used in the proofs form polynomially solvable sub-problems of MAX 3-SAT, 0–1 knapsack and TSP. Therefore, the results listed in the first paragraph also hold for polynomially solvable sub-problems of MAX 3-SAT, 0–1 knapsack and TSP. For example, there exists a polynomially solvable sub-problem of MAX 3-SAT such that data-independent neighborhood functions with the smooth property must contain all possible solutions for instances with $n \geqslant 4$ Boolean variables.

A *neighborhood function* for problem Π in NP optimization (NPO) [3] is a rule that maps an instance and feasible solution pair $(I, \boldsymbol{x})$, where $I \in D$ and $\boldsymbol{x} \in \mathrm{SOL}(I)$, to a set of feasible solutions. Therefore, a neighborhood function η for problem Π satisfies $\eta(I, \boldsymbol{x}) \subseteq \mathrm{SOL}(I)$ for every instance $I \in D$ and every solution $\boldsymbol{x} \in \mathrm{SOL}(I)$. Given an instance and feasible solution pair $(I, \boldsymbol{x})$, where $I \in D$ and $\boldsymbol{x} \in \mathrm{SOL}(I)$, $\eta(I, \boldsymbol{x})$ is referred to as the neighborhood of solution $\boldsymbol{x}$. In this paper, a solution is not permitted to be a member of its own neighborhood (i.e., $\boldsymbol{x} \notin \eta(I, \boldsymbol{x})$ for all instances $I \in D$ and solutions $\boldsymbol{x} \in \mathrm{SOL}(I)$). This restriction is consistent and compatible with the local search algorithm formulation.

To characterize properties of neighborhood functions, the following definitions are needed. Define the *size of a neighborhood function* η for an instance I to be $\max_{\boldsymbol{x} \in \mathrm{SOL}(I)} |\eta(I, \boldsymbol{x})|$. A neighborhood function η for Π is *complete* if $\eta(I, \boldsymbol{x}) = \mathrm{SOL}(I) - \{\boldsymbol{x}\}$ for every instance $I \in D$ (with length[I] sufficiently large, since the size of the neighborhood function is analyzed asymptotically) and $\boldsymbol{x} \in \mathrm{SOL}(I)$. A neighborhood function in which all local optima are global optima is said to have the *global search* (GS) *property*. A neighborhood function in which all strict local optima are global optima is said to have the *smooth property*. Suppose that for every solution if the neighborhood function η can be searched in polynomial time for an improving solution or else $\boldsymbol{x}$ is deemed a local optimum, then η is said to be *polynomially computable*.