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# Discounting Infinite Games But How and Why? [1]

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**Abstract**

In a recent paper de Alfaro, Henzinger and Majumdar [8] observed that discounting successive payments, the procedure that is employed in the classical stochastic game theory since the seminal paper of Shapley [16], is also pertinent in the context of much more recent theory of stochastic parity games [7,6,5] which were proposed as a tool for verification of probabilistic systems.
We show that, surprisingly perhaps, the particular discounting used in [8] is in fact very close to the original ideas of Shapley. This observation allows to realize that the specific discounting of [8] suffers in fact from some needless restrictions. We advocate that dropping the constraints imposed in [8] leads to a more general and elegant theory that includes parity and mean payoff games as particular limit cases.

*Keywords:* parity games, discounting games

## 1 Stochastic Games

The proper framework for our presentation are stochastic games introduced by Shapley [16].

Such games are played by two players [4] : the player 0 and the player 1. We

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[4] We consider here exclusively two players' zero sum games even if some definitions can obviously be stated in the broader framework of many players non zero sum games.


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are given a finite set [5] of *states* $S$, for each state $s \in S$ we have two finite sets of *actions* : $A(s)$ – the actions of player 0 and $B(s)$ the set of actions of player 1. If the system is at the state $s \in S$ both players choose simultaneously and independently actions $a \in A(s)$ and $b \in B(s)$ respectively and the system goes to a new state $s'$ with the probability $p(s' \mid s, a, b)$ that, as we can see, depends on the current state and the chosen actions. We suppose that the conditional probabilities are correctly and consistently defined, i.e., $0 \leq p(s' \mid s, a, b) \leq 1$ and $\sum_{s' \in S} p(s' \mid s, a, b) = 1$.

A *play* in such a game is an infinite sequence

$$p = (s_0, a_0, b_0), (s_1, a_1, b_1), (s_2, a_2, b_2), \ldots$$

of triples $(s_i, a_i, b_i)$ belonging to the set

$$T = \{(s, a, b) \mid s \in S \text{ and } a \in A(s), b \in B(s)\ \}$$

whose elements will be called *transitions.* Intuitively, the play $p$ describes the sequence of the visited states and the actions chosen by both players at each stage $i$ of the game.

A payoff mapping $u$ maps each possible play $p$ to a real number $u(p)$ — the payment received by player 0 from player 1 resulting from the play $p$. The obvious aim of 0 is to play in a way that maximizes his gain while player 1 tries to minimize his loss. Both players use strategies, that indicate how they should play at each stage of a game, i.e., which available action will be chosen. In general the choice of the next action can depend on the past history and can be probabilistic in nature, i.e., strategies provide a conditional probability distribution over the actions that are available at the current stage, see any of the following textbooks and monographs [18,10,19,17] for a formal definition. Fixing the strategies $\sigma$ of player 0 and $\tau$ of player 1 and an initial state $s$ yields a unique probability measure $\mu_{s,\sigma,\tau}$ over the Borel sets of plays starting at $s$. Now we can state more formally that the aim of player 0 is to choose, if possible, a strategy maximizing his expected payment

$$\mathbb{E}_{s,\sigma,\tau}(u) = \int u(p) \mu_{s,\sigma,\tau}(dp)$$

where the integral is taken over the set of all plays $p$ starting at $s$ (we assume tacitly that $u$ is integrable).

Varying the payment mapping $u$ we obtain different classes of stochastic games.

[5] Finiteness of the state space is not really necessary.