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Paired 2-disjoint path covers of multidimensional torus networks with faulty edges ☆

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ABSTRACT

A paired k-disjoint path cover (paired k-DPC for short) of a graph is a set of k disjoint paths joining k distinct source-sink pairs that cover all vertices of the graph. Clearly, the paired k-DPC is stronger than Hamiltonian-connectivity. The n-dimensional torus $T(k_1, k_2, \dots, k_n)$ (including the k-ary n-cube Q_n^k) is one of the most popular interconnection networks. In this paper, we obtain the following results. (1) Assume even $k_i \geq 4$ for $i = 1, 2, \dots, n$. Let $T = T(k_1, k_2, \dots, k_n)$ be a bipartite torus and F be a set of faulty edges with $|F| \leq 2n - 3$. Given any four vertices s_1, t_1, s_2 and t_2, such that each partite set contains two vertices. Then the graph $T - F$ has a paired 2-DPC consisting of $s_1 - t_1$ path and $s_2 - t_2$ path. And the upper bound $2n - 3$ of edge faults tolerated is optimal. The result is a generalization of the result of Park et al. concerning the case of $n = 2$ [17]. (2) Assume $k_i \geq 3$ for $i = 1, 2, \dots, n$, with at most one k_i being even. Let $T = T(k_1, k_2, \dots, k_n)$ be a torus and F be a set of faulty edges with $|F| \leq 2n - 4$. Then the graph $T - F$ has a paired 2-DPC. And the upper bound $2n - 4$ of edge faults tolerated is nearly optimal. The result is a generalization of the result of Park concerning the case of $n = 2$ [16]. Our brief proofs are based on a technique that is of interest and may find some applications.

1. Introduction

Hamiltonian property is one of the major requirements in designing network topologies since a topology structure containing Hamiltonian paths or cycles can efficiently simulate algorithms designed on linear arrays or rings. Element (vertex and/or edge) failure is inevitable when a large parallel computer system is put in use. In this regard, the fault-tolerant capacity of a network is a critical issue in parallel computing. There is a large of literature on (fault-tolerant) path and/or cycle embedding of various interconnection networks.

Given any two disjoint sets of k labeled vertices $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$ in a graph G, called sources and sinks, respectively. If there exist k disjoint paths $P_1, P_2, \dots, P_k$ in G, where P_i joins s_i and t_i for $i = 1, 2, \dots, k$, and they cover all vertices of G, then G is said to have a paired many-to-many k-disjoint path cover (paired k-DPC for short). It is easy to show that a paired k-DPC implies a paired s-DPC for $s = 1, 2, \dots, k$. Therefore, the paired many-to-many k-disjoint path cover is stronger than Hamiltonian-connectivity. For $k \geq 2$, the problem of the paired k-DPC has been investigated for hypercubes [4,5,7,8,10] and other classes of interconnection networks [11,13,16–18]. Relative problem of an unpaired k-DPC of interconnection networks has also been investigated [2,3,22].

The n-dimensional torus $T(k_1, k_2, \dots, k_2)$ (including the k-ary n-cube) is one of the most popular interconnection networks, it has many excellent topological properties. The (fault-tolerant) path and/or cycle embedding of tori [6,12,21] and k-ary n-cubes [9,14,15,19,20] has been extensively studied.

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In this paper, we investigate the problem of paired 2-DPC of the multidimensional torus network with faulty edges and obtain the following results.

Theorem 1. *Assume even $k_i \geq 4$ for $i = 1, 2, \ldots, n$. Let $T = T(k_1, k_2, \ldots, k_n)$ be a bipartite n-dimensional torus and F be a set of faulty edges with $|F| \leq 2n - 3$. Given any four vertices s_1, t_1, s_2 and t_2, such that each partite set contains two vertices, then the graph $T - F$ has a paired 2-DPC consisting of $s_1 - t_1$ path and $s_2 - t_2$ path. And the upper bound $2n - 3$ of edge faults tolerated is optimal.*

Theorem 2. *Assume $k_i \geq 3$ for $i = 1, 2, \ldots, n$, with at most one k_i being even. Let $T = T(k_1, k_2, \ldots, k_n)$ be a non-bipartite n-dimensional torus and F be a set of faulty edges with $|F| \leq 2n - 4$. Given any four vertices s_1, t_1, s_2 and t_2, then the graph $T - F$ has a paired 2-DPC consisting of $s_1 - t_1$ path and $s_2 - t_2$ path. The upper bound $2n - 4$ of edge faults tolerated is nearly optimal.*

Our results generalize the results of Park et al. concerning the case of $n = 2$ (see Lemmas 2 and 3). In Section 5, we consider the problem of paired 2-DPC of general non-bipartite multidimensional tori with faulty edges and obtain a result (Theorem 4) similar to Theorem 2. Our brief proofs of the theorems are based on a technique (Theorem 3 in Section 3) that is of interest and may find some applications.

2. Preliminaries

The terminology and notation used in this paper follow [1]. As usual, the vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We use $P = (v_1, v_2, \ldots, v_k)$ to denote a path with k vertices, where two vertices v_1 and v_k are called its end-vertices, and P is also called a $v_1 - v_k$ path. We use C to denote a cycle with at least three vertices. A cycle (respectively, path) containing all vertices of a graph G is called a Hamiltonian cycle (respectively, Hamiltonian path) of G. A graph is bipartite if its vertex set has a bipartition. A necessary and sufficient condition for a graph to be bipartite is that it contains no cycle with odd vertices. Let $E' \subset E(G)$, the notation $G - E'$ denotes the subgraph obtained from G by removing all edges in E'.

Let $k_i \geq 3$ for $i = 1, 2, \ldots, n$. An n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ is a graph with $\prod_{i=1}^{n} k_i$ vertices, its any vertex v can be denoted by an n-tuple $v = (x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq k_i - 1$ for $i = 1, 2, \ldots, n$, and the vertex v is adjacent to exactly $2n$ vertices $(x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_n)$, where $x_i + 1$ and $x_i - 1$ are taken modulo k_i for $i = 1, 2, \ldots, n$. For any i, an edge between two vertices $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$ (or $(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n)$) is called an edge of dimension i, and the set of all i-dimensional edges is denoted by E_i for $i = 1, 2, \ldots, n$.

It is easy to see that $E(T) = \bigcup_{i=1}^{n} E_i$ and the torus $T = T(k_1, k_2, \ldots, k_n)$ is vertex-symmetric. If $k_1 = k_2 = \ldots = k_n = k$, then T is called a k-ary n-cube, and denoted by Q_n^k. Clearly, Q_n^k is also edge-symmetric.

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. The Cartesian product of two graphs G and H, denoted by $G \times H$, is the graph with $V(G \times H) = \{u_i v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$, and a vertex uv is adjacent to a vertex $u'v'$ if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.

One can recursively define the Cartesian product of n graphs. The Cartesian product of graphs satisfies the commutative law and the associative law.

It is easy to show that the n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ is isomorphic to $C_{k_1} \times C_{k_2} \times \ldots \times C_{k_n}$, where C_{k_i} is a cycle with $k_i (\geq 3)$ vertices for $i = 1, 2, \ldots, n$. Thus,

$$\begin{aligned} T(k_1, k_2, \ldots, k_n) &= T(k_1, k_2, \ldots, k_{n-1}) \times C_{k_n} \\ &= T(k_1, k_2, \ldots, k_{n-2}) \times T(k_{n-1}, k_n), \end{aligned}$$

this simple fact will be used in the proof of the theorems.

We give three lemmas as follows.

Lemma 1. *(See [12].) Let $T = T(k_1, k_2, \ldots, k_n)$ be an n-dimensional torus and F be a set of faulty edges with $|F| \leq 2n - 2$. Then the graph $T - F$ contains a Hamiltonian cycle.*

Lemma 2. *(See [17].) Assume even $k_i \geq 4$ for $i = 1, 2$. Let $T = T(k_1, k_2)$ be a bipartite torus, f be a faulty edge, and s_1, t_1, s_2 and t_2 be any four vertices, such that each partite set contains two vertices. Then the graph $T - f$ has a paired 2-DPC consisting of $s_1 - t_1$ path and $s_2 - t_2$ path.*

Lemma 3. *(See [16].) Assume $k_1 \geq 3$ and odd $k_2 \geq 3$. Let $T(k_1, k_2)$ be a non-bipartite torus, and s_1, t_1, s_2 and t_2 be any four vertices. Then the torus $T(k_1, k_2)$ has a paired 2-DPC consisting of $s_1 - t_1$ path and $s_2 - t_2$ path.*

3. Theorem 3 and its proof

We give the following Theorem 3 that is a main technique of our brief proofs of Theorems 1, 2 and 4, and it is of interest and may find some applications.

Theorem 3. *Assume G and H are two graphs with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, and $G \times H$ is the Cartesian product of G and H. Let $G^{(j)} = G \times v_j$ be a copy of G in $G \times H$ for $j = 1, 2, \ldots, n$, and $H^{(i)} = u_i \times H$ be a copy of H in $G \times H$ for $i = 1, 2, \ldots, m$. Assume F_0 and F_1 are two disjoint sets of faulty edges in the Cartesian product $G \times H$, such that $F_0 \subset \bigcup_{j=1}^{n} E(G^{(j)})$ and $F_1 \subset \bigcup_{i=1}^{m} E(H^{(i)})$.*

(1) If $f \in F_0$, we use f' to denote the corresponding edge of f in the graph $G^{(1)}$, and if $f \in F_1$, we use f' to denote the corresponding edge of f in the graph $H^{(1)}$. Let $F_0' = \{f' | f \in F_0\} \subset E(G^{(1)})$ and $F_1' = \{f' | f \in F_1\} \subset E(H^{(1)})$. Assume the graph $G^{(1)} - F_0'$ contains a subgraph G_0 and the graph $H^{(1)} - F_1'$ contains a subgraph H_0. Then the graph $G \times H - (F_0 \cup F_1)$ contains a subgraph $G_0 \times H_0$.

(2) Assume the graph G contains a subgraph G_0 by removing at most r_0 arbitrary edges, and the graph H contains a subgraph H_0 by removing at most r_1 arbitrary edges. If $|F_0| \leq r_0$ and $|F_1| \leq r_1$, then the graph $G \times H - (F_0 \cup F_1)$ contains a subgraph $G_0 \times H_0$.