Information Processing Letters 116 (2016) 157–162

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

Mahonian STAT on words

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ARTICLE INFO

Article history:
Received 19 August 2014
Accepted 2 September 2015
Available online 25 September 2015
Communicated by S.M. Yiu

Keywords:
Words and Permutations
Statistic
Pattern
Equidistribution
Combinatorial problems

ABSTRACT

In 2000, Babson and Steingrímsson introduced the notion of what is now known as a permutation vincular pattern, and based on it they re-defined known Mahonian statistics and introduced new ones, proving or conjecturing their Mahonity. These conjectures were proved by Foata and Zeilberger in 2001, and by Foata and Randrianarivony in 2006.
In 2010, Burstein refined some of these results by giving a bijection between permutations with a fixed value for the major index and those with the same value for STAT, where STAT is one of the statistics defined and proved to be Mahonian in the 2000 Babson and Steingrímsson's paper. Several other statistics are preserved as well by Burstein's bijection.
At the Formal Power Series and Algebraic Combinatorics Conference (FPSAC) in 2010, Burstein asked whether his bijection has other interesting properties. In this paper, we not only show that Burstein's bijection preserves the Eulerian statistic ides, but also use this fact, along with the bijection itself, to prove Mahonity of the statistic STAT on words we introduce in this paper. The words statistic STAT introduced by us here addresses a natural question on existence of a Mahonian words analogue of STAT on permutations. While proving Mahonity of our STAT on words, we prove a more general joint equidistribution result involving two six-tuples of statistics on (dense) words, where Burstein's bijection plays an important role.

1. Introduction

In [1], the notion of what is now known as a *vincular pattern*[1] on permutations was introduced, and it was shown that almost all known *Mahonian* permutation statistics (that is, those statistics that are distributed as INV or as MAJ to be defined in Section 2) can be expressed as combinations of vincular patterns. The authors of [1] also introduced some new vincular pattern-based permutation statistics, showing that some of them are Mahonian and conjecturing that others are Mahonian as well. These conjectures were proved later in [4,5], and recently, alternative proofs based on *Lehmer code transforms* were given in [8].

Three statistics expressed in terms of vincular pattern combinations in [1] (namely, MAK, MAD and DEN) are known to be Mahonian not only on permutations, but also on words (see [3, Theorem 5]); more precisely, for any word v, the three statistics are distributed as INV on the set of rearrangements of the letters of v.

One of the statistics defined and shown to be Mahonian in [1] is STAT. Generalizing a result in [4], Burstein [2] shown the equidistribution of STAT and MAJ together with other statistics by means of an involution p on the set of permutations. At the Formal Power Series and Algebraic Combinatorics Conference (FPSAC) in 2010, Burstein asked whether p has other interesting properties.

In this paper, we not only show that p preserves the Eulerian statistic ides (which is not preserved, e.g. by the

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[1] Such patterns are called *generalized patterns* in [1].

http://dx.doi.org/10.1016/j.ipl.2015.09.006
0020-0190/

bijection Φ on words [3] mapping MAD to INV), but also use this fact, along with p itself, to prove Mahonity of the statistic STAT on words introduced in Subsection 2.2 (see relation (3)). The words statistic STAT introduced by us in this paper addresses a natural question on existence of a Mahonian words analogue of STAT on permutations. While proving Mahonity of our STAT on words, we prove a more general joint equidistribution result involving two six-tuples of statistics on (dense) words, where the bijection p plays an important role (see Theorems 1 and 2 in Section 5).

2. Preliminaries

We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, by $\mathfrak{S}_n$ the set of permutations of $[n]$, and by $[q]^n$ the set of length n words over the alphabet $[q]$. Clearly $\mathfrak{S}_n \subset [q]^n$ for $q \geq n > 1$. A word v in $[q]^n$ is said to be *dense* if each letter in $[q]$ occurs at least once in v. Dense words are also called *multi-permutations*.

2.1. Statistics

A *statistic* on $[q]^n$ (and thus on $\mathfrak{S}_n$) is an association of an integer to each word in $[q]^n$. Classical examples of statistics are:

$$\mathsf{INV}\, v = \operatorname{card}\{(i, j)\ :\ 1 \leq i < j \leq n, v_i > v_j\},$$

$$\mathsf{MAJ}\, v = \sum_{\substack{1 \leq i < n \\ v_i > v_{i+1}}} i,$$

$$\mathsf{des}\, v = \operatorname{card}\{i\ :\ 1 \leq i < n, v_i > v_{i+1}\},$$

where $v = v_1 v_2 \ldots v_n$ is a length n word. For example, $\mathsf{INV}(31\,425) = 3$, $\mathsf{MAJ}(3\,314\,452) = 8$, and $\mathsf{des}(8\,416\,422) = 4$.

For a word v and a letter a in v, other than the largest one in v, let us denote by $next_v(a)$ the smallest letter in v larger than a. With this notation, we define

$$\mathsf{ides}\, v = \operatorname{card}\{a : \text{ there are } i \text{ and } j, 1 \leq i < j \leq n, \text{ with } \\ v_i = next_v(a) \text{ and } v_j = a\}.$$

Clearly, when v is a permutation, $\mathsf{ides}\, v$ is simply $\mathsf{des}\, v^{-1}$, where v^{-1} is the inverse of v. For example, $\mathsf{ides}(144\,625) = 2$, and the corresponding values for a are 2 and 5.

For a set of words S, two statistics ST and ST′ have the same distribution (or are equidistributed) on S if, for any k,

$$\operatorname{card}\{v \in S : \mathsf{ST}\, v = k\} = \operatorname{card}\{v \in S : \mathsf{ST}'\, v = k\},$$

and it is well-known that INV and MAJ have the same distribution on both, the set of permutations and that of words.

A *multi-statistic* is simply a tuple of statistics.

2.2. Vincular patterns

Let $1 \leq r \leq q$ and $1 \leq m \leq n$, and let $v \in [r]^m$ be a dense word. One says that v occurs as a (classical) pattern in $w = w_1 w_2 \cdots w_n \in [q]^n$ if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $w_{i_1} w_{i_2} \cdots w_{i_m}$ is order-isomorphic to v. For example, 1231 occurs as a pattern in 6 214 562, and the three occurrences of it are 2452, 2462 and 2562.

Vincular patterns were introduced in the context of permutations in [1] and they were extensively studied since then (see Chapter 7 in [6] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:

- Any pair of two adjacent letters may now be underlined, which means that the corresponding letters in the permutation must be adjacent.[2] For example, the pattern $2\,\underline{1\,3}$ occurs in the permutation 425 163 four times, namely, as the subsequences 425, 416, 216 and 516. Note that, the subsequences 426 and 213 are *not* occurrences of the pattern because their last two letters are not adjacent in the permutation.
- If a pattern begins (resp., ends) with a hook[3] then its occurrence is required to begin (resp., end) with the leftmost (resp., rightmost) letter in the permutation. For example, there are two occurrences of the pattern $\llcorner 2\,\underline{1\,3}$ in the permutation 425 163, which are the subsequences 425 and 416.

The notion of a vincular pattern is naturally extended to words. For example, in the word 6 214 562, 645 is an occurrence of the pattern $3\,\underline{1\,2}$, and 262 is that of $1\,\underline{2\,1}\lrcorner$.

For a set of patterns $\{p_1, p_2, \ldots\}$ we denote by $(p_1 + p_2 + \ldots)$ the statistic giving the total number of occurrences of the patterns in a permutation. It follows from definitions that

$$\mathsf{MAJ}\, v = (1\,\underline{3\,2} + 1\,\underline{2\,1} + 2\,\underline{3\,1} + 2\,\underline{2\,1} + 3\,\underline{2\,1} + \underline{2\,1})\, v. \quad (1)$$

A vincular pattern of the form $u\,\underline{v\,x}$, with $\{u, v, x\} = \{1, 2, 3\}$, is determined by the relative order of u, v and x. For example, $2\,\underline{1\,3}$ is determined by $v < u < x$, and $3\,\underline{2\,1}$ by $x < v < u$.

An *extension* of a vincular pattern $u\,\underline{v\,x}$, $\{u, v, x\} = \{1, 2, 3\}$, is the combination of the vincular patterns obtained by replacing an order relation involving u (possibly both of them if there are two) by its (their) weak counterpart. For example,

- the unique extension of $1\,\underline{3\,2}$ is $(1\,\underline{3\,2} + 1\,\underline{2\,1})$; and
- the three extensions of $2\,\underline{3\,1}$ are:
  $(2\,\underline{3\,1} + 1\,\underline{2\,1})$,
  $(2\,\underline{3\,1} + 2\,\underline{2\,1})$, and
  $(2\,\underline{3\,1} + 1\,\underline{2\,1} + 2\,\underline{2\,1})$.

[2] The original notation for vincular patterns uses dashes: the absence of a dash between two letters of a pattern means that these letters are adjacent in the permutation.

[3] In the original notation the role of hooks was played by square brackets.