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On triangulation axes of polygons ☆

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ABSTRACT

We propose the *triangulation axis* as an alternative skeletal structure for a simple polygon P. This axis is a straight-line tree that can be interpreted as an anisotropic medial axis of P, where inscribed disks are line segments or triangles. The underlying triangulation that specifies the anisotropy can be varied, to adapt the axis so as to reflect predominant geometrical and topological features of P. Triangulation axes typically have much fewer edges and branchings than the Euclidean medial axis or the straight skeleton of P. Still, they retain important properties, as for example the reconstructability of P from its skeleton. Triangulation axes can be computed from their defining triangulations in $O(n)$ time. We investigate the effect of using several optimal triangulations for P. In particular, careful edge flipping in the constrained Delaunay triangulation leads, in $O(n \log n)$ overall time, to an axis competitive to 'high quality axes' requiring $\Theta(n^3)$ time for optimization via dynamic programming.

1. Introduction

Let P be a simply connected and closed polygon in the plane. A circular disk $D \subset P$ is called *maximal* (for P) if there is no other disk $D' \subset P$ with $D' \supset D$. The (Euclidean) *medial axis* of P is the set of centers of all maximal disks for P. This tree-like skeletal structure has proved a very useful descriptor of shape. Applications in diverse areas exist, and various construction algorithms have been proposed; see e.g., [2,6,8] and references therein.

The medial axis is a unique structure, as is the so-called *straight skeleton* [3,8] of P, which is composed of angular bisectors of P and can serve as a piecewise-linear alternative to the medial axis. In certain applications, however, it is desirable to have some flexibility in designing a skeletal structure, be it for keeping its size small so as to reflect only the essential parts of P, or for the sake of stability with respect to slight boundary changes of P. Several attempts have been made to adapt and prune the medial axis and the straight skeleton accordingly; see Attali et al. [6] and Siddiqi and Pizer [16], and Tanase and Veltkamp [17], respectively.

In the present note we propose a different idea, namely, of putting some anisotropy on the polygon P. Distances are measured differently at different locations within P, by varying the shape of the inscribed disks. (Anisotropic *Voronoi diagrams* where distances are measured individually from each defining point site have been introduced in Labelle and Shewchuk [14].) We divide the polygon P into triangles, and allot to each triangle a continuous family of unit disks, resulting from appropriately defined convex distance functions. (Voronoi diagrams for *convex distance functions* have been considered first in Chew and Drysdale [11].) The resulting skeleton is a straight-line tree resembling (but not equaling) the dual graph of the chosen triangulation of P. It always consists of fewer edges than the medial axis or the straight skeleton. When using the various known types of triangulation (e.g., constrained

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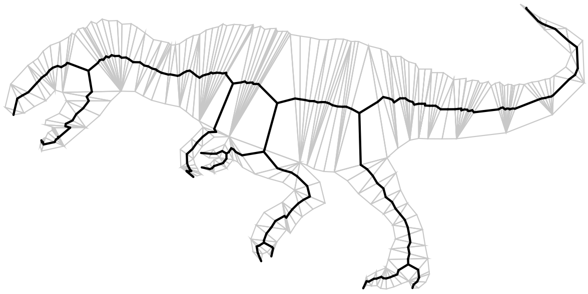

Fig. 1. A triangulation axis of a simple polygon.

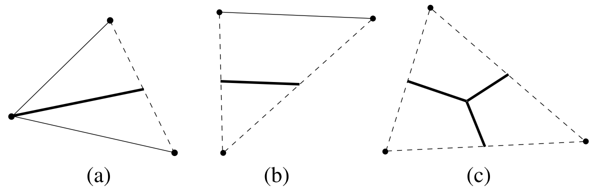

Fig. 2. Triangle types: (a) ear triangle, (b) link triangle, and (c) branch triangle. Diagonals of P are drawn in dashed style, and triangulation axis parts in bold style.

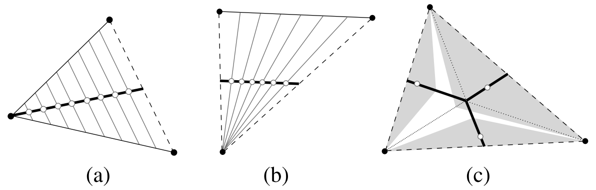

Fig. 3. Maximal disks for the anisotropic convex distance function within (a) an ear triangle, (b) a link triangle, and (c) a branch triangle. White dots mark the centers (centroids) of the shown disks.

Delaunay [15,10], minimum weight [13]), and also other triangulations optimal in different respects, we gain the needed flexibility, with the ultimate aim of defining a simple, stable, and characteristic skeletal axis structure for P.

In particular, we show that the constrained Delaunay triangulation of P, when post-processed by a small number of edge flips based on visibility within P, leads to satisfactory results: The empty-circle property ensures some closeness to the medial axis, and flipping has the effect of pruning away unimportant features. Several $O(n \log n)$ construction algorithms are available for this triangulation [8], so the new skeleton is fast and easier to compute than the medial axis or the straight skeleton.

A preliminary version of this work appeared in [4].

2. Triangulation axis

A *triangulation* T of a simple polygon P is a partition of P into triangles whose vertices are all from P. Let P have n vertices. We will assume $n \geq 4$ throughout, so that T contains at least one diagonal of P. To define what we will call the *triangulation axis*, $M_T(P)$, *of* P *and* T, the triangles which constitute T are categorized into three types: *ear* triangles, *link* triangles, and *branch* triangles – having one, two, or three sides that are diagonals of P, respectively; see Fig. 2.

Depending on its type, a triangle Δ contributes a specific part to $M_T(P)$. If Δ is an ear triangle, then its axis part is the line segment that connects the midpoint of its unique bounding diagonal d of P to the vertex of Δ opposite to d. If Δ is a link triangle, then it contributes to $M_T(P)$ the line segment connecting the midpoints of the two bounding diagonals of P. Finally, if Δ is a branch triangle, then the three line segments that connect its side midpoints to the centroid[1] of Δ are taken. See Fig. 2 again, where the individual axis parts are drawn in bold lines.

The triangulation axis $M_T(P)$ is now defined as the geometric graph that has the aforementioned line segments as its edges and their endpoints as its vertices. $M_T(P)$ is a (straight-line) tree, as can be shown by an easy induction argument.

A particular triangulation axis of a polygon is depicted in Fig. 1. Observe that link triangles (which typically constitute the majority in T) give rise to homothetic copies of P's boundary parts in the axis.

Indeed, $M_T(P)$ can be interpreted as an *anisotropic medial axis* of P. When suitable convex unit disks are used, the centroids of all possible *maximal* inscribed disks for P (as defined in Section 1) will delineate the triangulation axis. This is made explicit in Fig. 3. Maximal disks for ear triangles and link triangles are just line segments of varying slopes. These line segments are parallel to the unique bounding diagonal for an ear triangle, see (a), and fan out from a vertex for a link triangle, see (b). Maximal disks for a branch triangle are of triangular shape, see (c), based on a particular side of the branch triangle and having one vertex on the respective median line. Note that maximal disks (and thus the anisotropy they exert) change continuously when their centers are moved along $M_T(P)$.

Triangulation axes are quite natural skeletal structures for polygons, though as far as is known to the authors, they did not receive much attention in the literature. We found recent mention of a triangulation axis in Wang [18] for GIS applications, who refers to Ai and van Oosterom [1] for earlier use. No systematic study of $M_T(P)$ has been provided though, and only the constrained Delaunay triangulation [10] of P has been used for T. In the following sections, we will elaborate on some structural and algorithmic properties of triangulation axes, and give some experimental results that reflect the behavior of this structure in dependency of the underlying polygon triangulation.

3. Basic properties

A nice feature of triangulation axes is their small combinatorial size.

Lemma 1. *Any triangulation axis of a simple polygon P with n vertices has between $n - 2$ and $2n - 6$ edges.*

[1] Instead of the centroid, a different suitable point in Δ might be chosen; see Section 6.