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Approximation algorithms for optimization problems in graphs with superlogarithmic treewidth

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Abstract

We present a generic scheme for approximating NP-hard problems on graphs of treewidth $k = \omega(\log n)$. When a tree-decomposition of width ℓ is given, the scheme typically yields an $\ell/\log n$-approximation factor; otherwise, an extra $\log k$ factor is incurred. Our method applies to several basic subgraph and partitioning problems, including the *maximum independent set* problem.

Keywords: Approximation algorithms; NP-hard problems; Partial k-trees; Bounded treewidth

1. Introduction

One of the most successful parameterization of graphs is that of *treewidth*. While the formal definition is deferred to the next section, graphs of treewidth k, also known as *partial k-trees*, are graphs that admit a tree-like structure, known as their *tree-decomposition* of *width k*.

A wide variety of NP-hard graph problems have been shown to be solvable in polynomial time, or even linear time, when constrained to partial k-trees [2,3, 11]. For some of these problems polynomial time solutions are possible for graphs of treewidth $O(\log n)$ or $O(\log n/\log\log n)$ [3,11].

A standard example of a problem solvable in graphs of treewidth $O(\log n)$ is the maximum independent

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set (MIS) problem [3], which is that of finding a maximum collection of pairwise nonadjacent vertices. In the weighted version of the problem, vertices are given with weights and we seek an independent set of maximum total weight. For general graphs, the best polynomial-time approximation ratio known for MIS is $n\,(\log\log n)^2/\log^3 n$ [5]. On the other hand, it is known that unless NP $\subseteq$ ZPTIME$(2^{(\log n)^{O(1)}})$, no polynomial-time algorithm can achieve an approximation guarantee of $n^{1-O(1/(\log n)^{\gamma})}$ for some constant γ [8].

In this paper, we investigate the approximability status of some of the aforementioned NP-hard problems, where our main interest is in graphs of treewidth $k = \omega(\log n)$. We focus our study on MIS, deriving further applications of our method by extensions of that given for MIS.

Better approximation bounds for MIS are achievable for special classes of graphs. For the purposes of this paper, a class that properly contains partial k-trees is that of *k-inductive graphs*. A graph is said to be k-inductive if there is an ordering of its vertices so that each vertex has at most k higher-numbered neighbors. If such an ordering exists, it can be found by iteratively choosing and removing a vertex of minimum degree in the remaining graph. From this definition, it is clear that k-trees, and thus also partial k-trees, are k-inductive. A k-inductive graph is easily $(k+1)$-colored by processing the vertices in their reverse inductive order, assigning each vertex one of the colors not used by its at most k previously colored neighbors. This implies that the largest weight color class approximates the weighted MIS within a factor of $k+1$. The best approximation known for MIS (and weighted MIS) in k-inductive graphs is $O(k \log\log k/\log k)$ [7].

1.1. New contribution

We present a novel generic scheme for approximation algorithms for maximum independent set and other NP-hard graph optimization problems constrained to graphs of treewidth $k = \omega(\log n)$. Our scheme leads to deterministic polynomial-time algorithms that achieve an approximation ratio of $\ell/\log n$ when a tree-decomposition of width $\ell = \Omega(\log n)$ is given.

Our scheme can be applied to any problem of finding a maximum induced subgraph with hereditary property Π and any problem of finding a minimum partition into induced subgraphs with hereditary property Π provided that for graphs with given tree-decomposition of logarithmic or near logarithmic width it can be solved exactly in polynomial time. All these approximation factors achievable in polynomial time are the best known for the aforementioned problems for graphs of superlogarithmic treewidth.

In case a tree-decomposition of width $\ell = k$ is not given, the approximation achieved by our method increases by a factor of $O(\log k)$.

2. Preliminaries

The notion of *treewidth* of a graph was originally introduced by Robertson and Seymour [10] in their seminal graph minors project. It has turned out to be equivalent to several other interesting graph theoretic notions, e.g., that of partial k-trees.

Definition 1. A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, where $\{X_i \mid i \in I\}$ is a collection of subsets of V, and $T = (I, F)$ is a tree, such that the following three conditions are fulfilled:

(1) $\bigcup_{i \in I} X_i = V$,
(2) for each edge $(v, w) \in E$, there exists a node $i \in I$, with $v, w \in X_i$, and
(3) for each vertex $v \in V$, the subgraph of T, induced by the nodes $\{i \in I \mid v \in X_i\}$ is connected.

The *size of* T is the number of nodes in T, that is, $|I|$. Each set X_i, $i \in I$, is called the *bag* associated with the ith node of T. The width of a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The *treewidth* of a graph is the minimum width of its tree-decomposition taken over all possible tree-decompositions of the graph.

It is well known that a graph G is a partial k-tree iff the treewidth of G is at most k [2]. For a graph with n vertices and treewidth k, a tree decomposition of width k can be found in time $O(n\,2^{O(k^3)})$ [4], whereas a tree decomposition of width $O(k \log k)$ and size $O(n)$ can be found in time polynomial in n [1].