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The domination number of exchanged hypercubes

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ABSTRACT

Exchanged hypercubes (Loh et al., 2005 [13]) are spanning subgraphs of hypercubes with about one half of their edges but still with many desirable properties of hypercubes. Lower and upper bounds on the domination number of exchanged hypercubes are proved which in particular imply that $\gamma(EH(2,t)) = 2^{t+1}$ holds for any $t \geqslant 2$. Using Hamming codes we also prove that $\gamma(EH(s,2^k-1)) \leqslant (2^s-2^k)\gamma(Q_{2^k-1}) + 2^{2^k-1}(\gamma(Q_s^-)+1)$ holds for $s \geqslant k \geqslant 3$.

1. Introduction

Hypercubes form a fundamental model for parallel computers and interconnection networks, cf. [22, Chapter 7]. They have many fine properties that are essential for network efficiency, such as recursive decomposition, lots of symmetries, low regularity, and small diameter. Hypercubes also allow straightforward (local) routing and are Hamiltonian. For more information on their fault tolerance with respect to the Hamiltonicity see [19,20] and the references therein. Having all this in mind it comes with no big surprise that machines based on hypercubes have actually been implemented, see [22, p. 115] for the list of implementations.

Interconnection networks often require a distribution of limited supply of resources and from this point of view various kinds of dominating sets serve as possible locations for placement of resources. For general aspects of the role of domination in complex networks see the book chapter [1]. Unfortunately, the exact domination number is known only for small dimensional hypercubes and two infinite families: $\gamma(Q_3)=2$, $\gamma(Q_4)=4$, $\gamma(Q_5)=7$, $\gamma(Q_6)=12$, and $\gamma(Q_n)=2^{n-k}$ for $n=2^k-1$ or $n=2^k$, see [8]. In general, $\gamma(Q_n) \leqslant 2^{n-3}$ for $n \geqslant 7$ [3]. For some variations of domination studied on hypercubes see [3,7,17], while for domination of closely related Fibonacci cubes see [4,18]. Domination was also studied on other types of interconnection networks as for instance on toroidal meshes [21].

Since domination is very difficult on hypercubes, they are not very appropriate when dealing with domination-type problems. In this note we instead study the domination number of exchanged hypercubes $EH(s,t)$. This two-parametric family of graphs was proposed by Loh et al. [13] and constitute a variation of the hypercube networks with numerous appealing properties, see [15] for their bipancyclicity and [10,14,16] for their connectivity and super connectivity, important measures for the fault-tolerance of networks. In the special case when $s=t$, the exchanged hypercubes coincide with the so-called dual-cubes, a class of hypercube-like networks studied in [2,5,11,12].

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We proceed as follows. In the next section we introduce the exchanged hypercubes, recall some of their properties, and define other concepts used in this note. Then, in Section 3, our results are presented. We prove several bounds on the domination number of exchanged hypercubes and deduce from them that if $t \geqslant 2$, then $\gamma(EH(2,t)) = 2^{t+1}$. This exact result appears appealing because, as we have noted above, the domination number of the usual hypercubes is an intrinsically difficult problem. Using the fact that Q_{2^k-1} contains a perfect code (which is just a corresponding Hamming code) we also prove that $\gamma(EH(s, 2^k - 1)) \leqslant (2^s - 2^k)\gamma(Q_{2^k-1}) + 2^{2^k-1}(\gamma(Q_s^-) + 1)$ holds for $s \geqslant k \geqslant 3$.

2. Preliminaries

Graphs considered here are simple, finite, and connected.

If n is a positive integer, then the n-dimensional *hypercube* (or *n-cube*, for short) Q_n is the graph with vertex set $\{0,1\}^n$, two vertices (strings) being adjacent if they differ in exactly one coordinate. Hypercubes are vertex-transitive graphs, hence all vertex-deleted subgraphs $Q_n - v$, $v \in V(Q_n)$, are isomorphic, we denote it with Q_n^-. The distance between vertices $u, v \in V(Q_n)$ is equal to the *Hamming distance* between u and v, denoted $H(u,v)$, that is, the number of coordinates in which u and v differ.

Exchanged hypercubes are spanning subgraphs of hypercubes. Let $u = u_{d-1} \ldots u_0 \in \{0,1\}^d$ be a binary string, $d \geqslant 1$. If $j \geqslant i$, then we will use the notation $u_{j:i}$ for the substring of u between u_j and u_i, that is, $u_{j:i} = u_j \ldots u_i$. For any integers $s \geqslant 1$ and $t \geqslant 1$, the *exchanged hypercube* $EH(s,t)$ is the graph with the vertex set $\{0,1\}^{s+t+1}$. Hence, if $u \in V(EH(s,t))$, then its coordinates are $u_{s+t} \ldots u_{t+1} u_t \ldots u_1 u_0$. Vertices u and v are adjacent if one of the following conditions is satisfied:

(i) $u_{s+t:1} = v_{s+t:1}$, $u_0 \neq v_0$,
(ii) $u_0 = v_0 = 1$, $H(u_{t:1}, v_{t:1}) = 1$, and $u_{s+t:t+1} = v_{s+t:t+1}$,
(iii) $u_0 = v_0 = 0$, $H(u_{s+t:t+1}, v_{s+t:t+1}) = 1$, and $u_{t:1} = v_{t:1}$.

Clearly, $EH(s,t)$ has 2^{s+t+1} vertices. If $u \in V(EH(s,t))$ and $u_0 = 0$, then the degree of u is $s+1$, otherwise the degree of u is $t+1$. It is also straightforward that for any s and t, the exchanged hypercube $EH(s,t)$ is isomorphic to $EH(t,s)$. The ratio of the number of edges in $EH(s,t)$ to that of Q_{s+t+1} is $1/2 + 1/(2(s+t+1))$ [6].

If G is a graph, then $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) - D$ is adjacent to some vertex of D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of G is a *perfect code* if any two vertices from D are at distance at least 3. Hence the closed neighborhoods of the vertices from a perfect code D partition the vertex of G, cf. [9, Theorem 4.1].

A *matching* of a graph G is a set of independent edges and a *perfect matching* is a matching M such that each vertex is an endpoint of an edge from M. Finally, if $X \subseteq V(G)$, then the closed neighborhood $N[X]$ is $\bigcup_{u \in X} N[u]$, where $N[u]$ is the closed neighborhood of u.

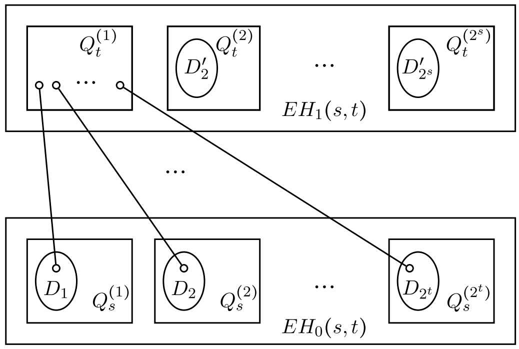

Fig. 1. Subgraphs $EH_0(s,t)$ and $EH_1(s,t)$ of $EH(s,t)$.

3. Results

We begin with the following bounds:

Theorem 3.1. *If $s, t \geqslant 1$ and $s \leqslant t$, then*

$$\max\{2^t\gamma(Q_s), 2^s\gamma(Q_t)\} \leqslant \gamma\big(EH(s,t)\big) \leqslant \big(2^s - 1\big)\gamma(Q_t) + 2^t\gamma(Q_s).$$

Proof. Consider the following edge-subsets of $EH(s,t)$:

$$E_1 = \{uv\colon u_{s+t:1} = v_{s+t:1},\ u_0 \neq v_0\},$$
$$E_2 = \big\{uv\colon u_{s+t:t+1} = v_{s+t:t+1},\ H(u_{t:1}, v_{t:1}) = 1,\ u_0 = v_0 = 1\big\},$$
$$E_3 = \big\{uv\colon u_{t:1} = v_{t:1},\ H(u_{s+t:t+1}, v_{s+t:t+1}) = 1,\ u_0 = v_0 = 0\big\}.$$

Let $EH_1(s,t)$ be the subgraph of $EH(s,t)$ induced by the edges E_2. Then $EH_1(s,t)$ is the disjoint union of 2^s copies of Q_t, we denote these cubes with $Q_t^{(i)}$, $1 \leqslant i \leqslant 2^s$. Indeed, fixing the leftmost s bits and fixing the rightmost bit to 1, the induced subgraph is isomorphic to Q_t. Moreover, there are no edges between two such induced subgraphs isomorphic to Q_t. Similarly, the subgraph $EH_0(s,t)$ of $EH(s,t)$ induced by the edges E_3 consists of 2^t subgraphs isomorphic to Q_s denoted with $Q_s^{(j)}$, $1 \leqslant j \leqslant 2^t$. Finally, the edges from E_1 form a perfect matching of $EH(s,t)$, it is a matching between $EH_0(s,t)$ and $EH_1(s,t)$. More precisely, for any i, any vertex of $Q_t^{(i)}$ has exactly one neighbor in $EH_0(s,t)$, each of these neighbors belonging to different $Q_s^{(j)}$. See Fig. 1.

For the upper bound, consider the t-cube $Q_t^{(1)}$. Then each of $Q_s^{(i)}$, $1 \leqslant i \leqslant 2^t$, has a (unique) neighbor in $Q_t^{(1)}$. In each of the cubes $Q_s^{(i)}$ select a minimum dominating set D_i such that if $x \in N[V(Q_t^{(1)})] \cap Q_s^{(i)}$ then $x \in D_i$. (Such a dominating set exists since hypercubes are vertex-transitive graphs.) Then $Q_t^{(1)}$ is dominated by $\bigcup_{i=1}^{2^t} D_i$, see Fig. 1 again. For $2 \leqslant i \leqslant 2^s$ let D_i' be a minimum dominating set of $Q_t^{(i)}$. Then $D = (\bigcup_{i=1}^{2^t} D_i) \cup (\bigcup_{i=2}^{2^s} D_i')$ is a dominating set of $EH(s,t)$. Clearly, $|D| = 2^t\gamma(Q_s) + (2^s - 1)\gamma(Q_t)$. The upper bound is proved.