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The computational complexity of the backbone coloring problem for bounded-degree graphs with connected backbones ☆

Robert Janczewski *, Krzysztof Turowski

Department of Algorithms and Systems Modelling, Gdańsk University of Technology, Narutowicza 11/12, Gdańsk, Poland

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ABSTRACT

Given a graph G, a spanning subgraph H of G and an integer $\lambda \geq 2$, a λ-backbone coloring of G with backbone H is a proper vertex coloring of G using colors $1, 2, \ldots,$ in which the color difference between vertices adjacent in H is greater than or equal to λ. The backbone coloring problem is that of finding such a coloring whose maximum color does not exceed a given limit k. In this paper, we study the backbone coloring problem for bounded-degree graphs with connected backbones and we give a complete computational complexity classification of this problem. We present a polynomial algorithm for optimal backbone coloring for subcubic graphs with arbitrary backbones. We also prove that the backbone coloring problem for graphs with arbitrary backbones and with fixed maximum degree (at least 4) is NP-complete. Furthermore, we show that for the special case of graphs with fixed maximum degree at least 5 and $\lambda \geq 4$ the problem remains NP-complete even for spanning tree backbones.

1. Introduction

Coloring problems have been one of main areas of interest in graph theory since the middle of the 19th century. Whenever real-world situations can be modeled by graphs and one's aim is to partition the set of objects into pairwise disjoint subsets of non-conflicting objects, this can be viewed as a graph coloring problem. For this reason, graph coloring has found applications in many problems such as job scheduling, wavelength assignment and frequency assignment.

In [1], the backbone coloring problem was motivated and introduced in the context of a general framework of graph coloring problems. In the graph model of the radio network the vertices represent the transmitters, receivers or base stations. Two adjacent vertices denote possible interference between transmitters broadcasting on the same or similar frequency channels. Given the definition of interference, the requested level of interference and the notion of similarity between frequency channels, the goal is to find a coloring which minimizes the total frequency band (the number of frequency channels) used in the network.

A backbone coloring problem within the given framework was also introduced and motivated in [1]. We distinguish a certain substructure of adjacent transmitters (called the backbone) from the rest of the network as more crucial for communication. It could, for example, model the connections between cluster heads and the other sensors in the same cluster of a sensor network [2] or hot spots with very busy patterns of communication in a radio network [3].

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* Corresponding author.
E-mail addresses: skalar@eti.pg.gda.pl (R. Janczewski), Krzysztof.Turowski@eti.pg.gda.pl (K. Turowski).

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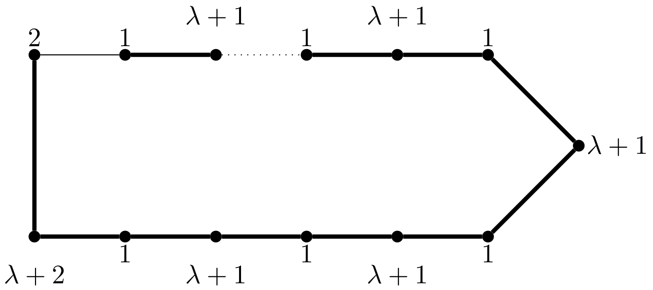

Fig. 1. Optimal backbone coloring of C_{2n+1} with backbone P_{2n+1}.

Formally, we define a λ-backbone coloring of a graph G and its spanning subgraph (backbone) H, where $\lambda \geq 2$ is an integer, to be a function $c: V(G) \to \mathbb{N}_+$ such that $c(u) \neq c(v)$ for each edge $uv \in E(G)$ and $|c(u) - c(v)| \geq \lambda$ for each edge $uv \in E(H)$. The minimum number k for which there exists a λ-backbone coloring of graph G with backbone H with $\max c(V(G)) = k$ is called the backbone chromatic number and denoted by $BBC_\lambda(G, H)$.

In this paper, we study the backbone coloring problem (given G with backbone H and integers k, λ, is there a λ-backbone coloring of G with backbone H with maximum color not exceeding k?) for bounded-degree graphs with connected backbones. This leads to a plausible interpretation in terms of radio networks since typically radio networks have rather low degree, but for every transmitter there is at least one other transmitter within its close range.

The remainder of the paper is organized as follows. In Section 2 we give some preliminary results. Next, in Sections 3 and 4, we show that the backbone coloring problem is polynomially solvable for graphs with maximum degree not exceeding 3. The last section contains proofs of the facts that this problem is NP-complete for graphs with fixed maximum degree ≥ 4 and for graphs with fixed maximum degree ≥ 5 and backbones being trees.

2. Preliminaries

The following basic observation turns out to be very useful for proving the properties of backbone coloring of low-degree graphs:

Theorem 1. *For any graph G with backbone H we have*

(i) $(\chi(H) - 1)\lambda + 1 \leq BBC_\lambda(G, H) \leq (\chi(G) - 1)\lambda + 1$,
(ii) *if* $\chi(G) = \chi(H)$ *then* $BBC_\lambda(G, H) = (\chi(G) - 1)\lambda + 1$.

Proof.

(i) The first inequality was originally proved in [6]. The second can be proved using the observation that for any coloring of G that uses colors $1, 2, \dots, \chi(G)$ we may substitute color i with color $(i-1)\lambda + 1$ and obtain a λ-backbone coloring of G with backbone H.
(ii) Trivial consequence of (i). □

From now on, we assume that G and H are fixed, connected and H is a spanning subgraph of G. We will write V instead of $V(G) = V(H)$.

3. $\Delta(G) \leq 2$

The following result shows that backbone coloring of graph G with $\Delta(G) \leq 2$ is easily solvable.

Theorem 2. *Let* $\Delta(G) \leq 2$. *Then*

(i) *if G has exactly one vertex then* $BBC_\lambda(G, H) = 1$,
(ii) *if G is bipartite with at least* 2 *vertices then* $BBC_\lambda(G, H) = \lambda + 1$,
(iii) *if G is an odd cycle and* $G \neq H$ *then* $BBC_\lambda(G, H) = \lambda + 2$,
(iv) *if G is an odd cycle and* $G = H$ *then* $BBC_\lambda(G, H) = 2\lambda + 1$.

Proof. Recall that in all cases we assume G is connected and H is connected and spanning subgraph of G.

(i) Trivial.
(ii) This was proved in [4].
(iii) In this case $G = C_{2n+1}$ and $H = P_{2n+1}$ for some n. In [4], it was shown that $BBC_\lambda(G, H) \leq \lambda + 1$ is possible only if G is bipartite. Therefore $BBC_\lambda(G, H) \geq \lambda + 2$. Fig. 1 shows how to color G with backbone H with $\lambda + 2$ colors (bold edges are in H).
(iv) Follows from Theorem 1(ii). □

4. $\Delta(G) = 3$

This case is still polynomial-time solvable but much harder. Our algorithm, which finds optimal backbone coloring (and therefore computes $BBC_\lambda(G, H)$) works in two steps. In the first step it computes an optimal coloring of G and the chromatic numbers $\chi(G)$ and $\chi(H)$. In the second, it applies one of the subalgorithms described in the next three subsections (the one that matches the computed values of $\chi(G)$ and $\chi(H)$). The first step can be done in $O(n^2)$ time since G and H are subcubic. The subalgorithms run in $O(n^2)$ time, so the whole process takes $O(n^2)$ time.

4.1. $\chi(G) \geq 4$

According to Brooks' theorem the only subcubic graph with $\chi(G) \geq 4$ is the complete graph K_4. In this case there are six possible spanning subgraphs (up to isomorphism),