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Polynomial-time algorithms for weighted efficient domination problems in AT-free graphs and dually chordal graphs

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ABSTRACT

An efficient dominating set (or perfect code) in a graph is a set of vertices the closed neighborhoods of which partition the vertex set of the graph. The minimum weight efficient domination problem is the problem of finding an efficient dominating set of minimum weight in a given vertex-weighted graph; the maximum weight efficient domination problem is defined similarly. We develop a framework for solving the weighted efficient domination problems based on a reduction to the maximum weight independent set problem in the square of the input graph. Using this approach, we improve on several previous results from the literature by deriving polynomial-time algorithms for the weighted efficient domination problems in the classes of dually chordal and AT-free graphs. In particular, this answers a question by Lu and Tang regarding the complexity of the minimum weight efficient domination problem in strongly chordal graphs.

1. Introduction

The concept of an efficient dominating set in a graph was introduced by Biggs [2] as a generalization of the notion of a perfect error-correcting code in coding theory. Given a (simple, finite, undirected) graph $G = (V, E)$, we say that a vertex *dominates* itself and each of its neighbors. An *efficient dominating set* in G is a subset of vertices $D \subseteq V$ such that every vertex $v \in V$ is dominated by precisely one vertex from D. Efficient domination has several interesting applications in coding theory and resource allocation of parallel processing systems [2,28,30]. The notion of an efficient dominating set appeared in the literature under various other names such as: *perfect code*, 1-*perfect code*, *independent perfect dominating set*, and *perfect dominating set*. Note, however, that the name *perfect dominating set* has also been used in the literature to denote a subset of vertices $D \subseteq V$ such that every vertex $v \in V \setminus D$ is dominated by precisely one vertex from D. See [32] for a nice historical overview of the notion of efficient dominating set.

A graph is *efficiently dominatable* if it contains an efficient dominating set. All paths are efficiently dominatable, and a cycle C_k on k vertices is efficiently dominatable if and only if k is a multiple of 3. Bange et al. [1] showed that if a graph G has an efficient dominating set, then all efficient dominating sets of G have the same cardinality, which equals the minimum cardinality of a dominating set of G. The efficient domination (ED) problem consists in determining whether the input graph is efficiently

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dominatable. The ED problem is NP-complete even for restricted graph classes such as planar cubic graphs [25], bipartite graphs [41], planar bipartite graphs [32], chordal bipartite graphs [32], chordal graphs [41], and line graphs of planar bipartite graphs of maximum degree three [5]. On the other hand, the ED problem is polynomial for several graph classes, including trees [1,17], block graphs [41], interval graphs [13,14,24,26], circular-arc graphs [13,24], cocomparability graphs [10,14], bipartite permutation graphs [32], permutation graphs [28], distance-hereditary graphs [32], trapezoid graphs [28,29], split graphs [12], dually chordal graphs [7], AT-free graphs [7,8], and hereditary efficiently dominatable graphs [34]. The efficient domination problem has also been studied from a parameterized point of view, see, e.g., [9,23].

In this paper, we consider two weighted versions of the ED problem. In its decision form, the minimization version of the problem can be stated as follows:

> Minimum Weight Efficient Dominating Set (Min-WED)
>
> *Input:* A graph G, vertex weights $w : V \to \mathbb{Z}$, an integer k.
>
> *Question:* Does G contain an efficient dominating set D of total weight $w(D) := \sum_{x \in D} w(x) \leq k$?

The maximization version of problem (Maximum Weight Efficient Dominating Set (Max-WED)), can be defined analogously, replacing the condition $w(D) \leq k$ with $w(D) \geq k$.

Clearly, a graph $G = (V, E)$ contains an efficient dominating set if and only if $(G, w, |V|)$ is a yes instance to the Min-WED problem, where $w(x) = 1$ for all $x \in V$. Consequently, the Min-WED problem is NP-complete in every class of graphs where the ED problem is NP-complete. On the other hand, the Min-WED problem is solvable in polynomial time for trees [40], cocomparability graphs [10,14], split graphs [12], interval graphs [13,14], circular-arc graphs [13], permutation graphs [28], trapezoid graphs [28,29], bipartite permutation graphs [32], convex bipartite graphs [7] and their superclass interval bigraphs [7], distance-hereditary graphs [32], block graphs [41] and hereditary efficiently dominatable graphs [16,34]. Since negative weights are allowed, the Max-WED problem is equivalent to the Min-WED problem.

We develop a framework for solving the Min-WED and Max-WED problems based on a reduction to the Maximum Weight Independent Set problem in the square of the input graph (this is done in Section 3). We then apply this framework, together with some existing results from the literature, to derive new polynomial cases of the Min-WED problems, namely the classes of dually chordal graphs and AT-free graphs (in Sections 4.1 and 4.2, respectively). The class of dually chordal graphs contains the class of strongly chordal graphs, for which the existence of a polynomial-time algorithm for the Min-WED problem was posed as an open problem by Lu and Tang in [32]. We give a linear-time algorithm for the Min-WED and Max-WED problems in the class of dually chordal graphs. Our algorithm for the Min-WED problem in the class of AT-free graphs is of complexity $\mathcal{O}(\min\{nm + n^2, n^\omega\})$, where $\omega < 2.3727$ is the matrix multiplication exponent [39], and n and m denote the number of vertices and edges of the input graph, respectively. This improves on the existing polynomial-time algorithms for the ED problem in AT-free graphs [7,8], both of which run in time $\mathcal{O}(n^4)$.

In Fig. 1, we show the Hasse diagram of the poset of most of the graph classes mentioned above, ordered with respect to inclusion. For each class, we state the complexity of the Min-WED problem, denoting by NP-c the fact that the problem is NP-complete in the corresponding class, while in the case of polynomial-time solvability, we state the running times of the fastest known algorithms. The inclusion relations in the figure were verified with help of the Information System on Graph Classes and their Inclusions [21].

2. Preliminaries

We only consider finite, simple and undirected graphs. As usual, the *neighborhood* of a vertex v in a graph $G = (V, E)$ is denoted by $N_G(v) := \{u \in V \mid uv \in E\}$ (or simply $N(v)$, if the graph is clear from the context), and the closed neighborhood of v is $N_G[v] := N_G(v) \cup \{v\}$ (or simply $N[v]$). The *degree* of a vertex v in a graph G is $\deg_G(v) := |N_G(v)|$. The *square* of a graph $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that $uv \in E^2$ if and only if either $uv \in E$ or u and v are distinct and have a common neighbor in G. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ such that two distinct vertices u and v of G are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. An *independent set* in a graph is a set of pairwise non-adjacent vertices, and a *clique* is a set of pairwise adjacent vertices. Given a graph G and a total ordering $\sigma = (v_1, \ldots, v_n)$ of its vertex set, we will denote by $G_{\sigma,i}$ the subgraph of G induced by $\{v_i, v_{i+1}, \ldots, v_n\}$, and by $N_{\sigma,i}(v)$ (resp. $N_{\sigma,i}[v]$), the neighborhood (resp., the closed neighborhood) of a vertex $v \in V(G_{\sigma,i})$ in $G_{\sigma,i}$.

Chordal graphs are graphs in which every cycle on at least 4 vertices has a chord (an edge connecting two non-consecutive vertices on the cycle). It is well known that chordal graphs are precisely the graphs that admit a perfect elimination ordering. A *perfect elimination ordering* of a graph G is a total ordering $\sigma = (v_1, \ldots, v_n)$ of its vertex set such that for every $i \in \{1, \ldots, n\}$, vertex v_i is simplicial in $G_{\sigma,i}$, that is, the set $N_{\sigma,i}(v_i)$ is a clique in $G_{\sigma,i}$ (equivalently: in G). *Dually chordal graphs* were introduced in [4] as graphs that admit a certain vertex ordering called a maximum neighborhood ordering. Given two vertices u and v in a graph $G = (V, E)$, vertex $u \in N[v]$ is a *maximum neighbor* of vertex v if $N[w] \subseteq N[u]$ holds for all $w \in N[v]$. A linear ordering $\sigma = (v_1, \ldots, v_n)$ of V is a *maximum neighborhood ordering* of G if for all $i \in \{1, \ldots, n\}$, vertex v_i has a maximum neighbor in the graph $G_{\sigma,i}$. Dually chordal graphs admit several equivalent characterizations and form a generalization of *strongly chordal graphs*, a well-known subclass of chordal graphs properly contain-