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A new lower bound for the number of perfect matchings of line graph

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ABSTRACT

Let $G=(V,E)$ be a graph, $V(G)=\{u_1,u_2,\cdots,u_n\}$ and $|E|$ be even. In this paper, we show that $M(L(G))\geq 2^{|E|-|V|+1}+\sum_{i=1}^{n}(f_G(u_i)+g_G(u_i))!!-n$, where $f_G(u_i)=d_G(u_i)-w(G-u_i)$, $w(G-u_i)$ is the number of components of $G-u_i$, $g_G(u_i)$ is the number of those components of $G-u_i$ each of which has an even number of edges, and $M(L(G))$ is the number of perfect matchings of the line graph $L(G)$. Also we show that $M(L(G))\geq \eta(\Delta)\cdot 2^{|E|-|V|-\Delta+2}$ for every 2-connected graph G, and give a sufficient and necessary condition about 2-connected graphs G such that $M(L(G))=\eta(\Delta)\cdot 2^{|E|-|V|-\Delta+2}$, where $\eta(\Delta)=\sum_{0\leq k\leq \Delta/2}\binom{\Delta}{2k}(2k)!!$, Δ is the maximum degree of G, and $(2k)!!=(2k-1)(2k-3)\cdots 3\cdot 1$.

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1. Introduction

The enumeration problems for maximum matchings and perfect matchings of a graph play an important role in graph theory and combinatorial optimization and have a wide applications in some field. For example, in the chemical field, the number of perfect matchings of bipartite graphs corresponds to Kekulé structures number (see [4]). In physical field, the Dimer problem is essentially equal to the number of perfect matchings of a graph (see [5]). The number of perfect matchings is an important topological index which had been applied for estimation of the resonant energy, total π-electron energy and calculation of Pauling bond order (see [6]). But the enumeration problem for perfect matchings in general graphs (even in bipartite graphs) is NP-hard (see [3,7]). So far, many mathematicians, physicists and chemists have given most of their attention to counting perfect matchings of graphs (see [1,4,8–10]).

Sumner [11] showed that every connected claw-free graph with an even number of vertices has a perfect matching. Since every line graph is claw-free, the line graph of a connected graph with an even number of edges has a perfect matching.

Dong and Yan showed that $M(L(G))\geq 2^{|E(G)|-|V(G)|+1}$ for every connected graph G, and also obtained the sufficient and necessary condition about the connected graphs G such that $M(L(G))=2^{|E(G)|-|V(G)|+1}$ (see [1]). In this paper, we show that $M(L(G))\geq 2^{|E|-|V|+1}+\sum_{i=1}^{n}(f_G(u_i)+g_G(u_i))!!-n$, and also give a sufficient and necessary condition about 2-connected graphs G such that $M(L(G))=\eta(\Delta)\cdot 2^{|E|-|V|-\Delta+2}$.

2. Definitions and preliminaries

The graphs considered in this paper are finite, undirected and simple graphs. For some notations and definitions undefined here, see [2].

Let $G=(V,E)$ be a graph, the degree of v is denoted by $d_G(v)$ for $v\in V$, E_v be the set of edges in G which are incident with v. Then $|E_v|=d_G(v)$. A vertex v is called a leaf of G if $d_G(v)=1$.

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The line graph of a graph G, denoted by $L(G)$, is defined as the graph with $V(L(G)) = E(G)$ such that any two vertices e and f of $L(G)$ are adjacent if e and f has a common end in G.

An edge subset $M \subseteq E(G)$ is a matching of G if no two edges in M are incident with a common vertex. A matching M of G is a perfect matching if every vertex of G is incident with an edge in M. The number of all perfect matchings of G is denoted by $M(G)$. It is obvious that if G is a complete graph with n vertices, where n is even, then $M(G) = (n-1)(n-3)\cdots 3\cdot 1$.

For any graph G, let $w(G)$ be the number of components of G, $p(G)$ be the number of those components of G each of which has an even number of edges. If G is a forest, $p(G)$ and $|V(G)|$ have the same parity. Thus, if G is a tree and $|V(G)|$ is odd, then $p(G-v)$ is even for all $v \in V(G)$.

Proposition 1. *(See [1].) Let T be a tree with $V(T) = \{v_1, v_2, \cdots, v_n\}$, where $n > 1$ is odd. Then*

$$M\big(L(T)\big) = \prod_{i=1}^{n} p(T-v_i)!!,$$

where $(2k)!! = (2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$ for any non-negative integer k, and $(-1)!! = 1$.

The next result follows immediately from Proposition 1.

Proposition 2. *(See [1].) Let T be a tree with n vertices, where $n > 1$ is odd. Then $M(L(T)) \geq 1$, where the equality holds if and only if $p(T-v) = 0$ or 2 for every $v \in V(T)$.*

Let G be a connected graph, $u \in V(G)$. We define that $f_G(u) = d_G(u) - w(G-u)$ and $g_G(u) = p(G-u)$. Then $f_G(u) \geq 0$, where the equality holds if and only if every edge of E_u is a bridge of G.

Proposition 3. *(See [1].) Let G be a connected graph with m edges. Then $f_G(u) + g_G(u) \equiv m (\mathrm{mod}\ 2)$ for each $u \in V(G)$.*

Let e be any edge of G with ends u and v. Let $G(u, w)$ be the graph obtained from $G-e$ by adding a new vertex w and adding a new edge joining w to u. $G(v, w)$ is defined similarly.

Proposition 4. *(See [1].) Let G be a graph and e be an edge of G with ends u and v. Then*

$$M\big(L(G)\big) = M\big(L\big(G(u,w)\big)\big) + M\big(L\big(G(v,w)\big)\big).$$

Let G be a connected graph with n vertices and m edges, T^* be a spanning tree of G, $E(G) - E(T^*) = \{e_1, e_2, \cdots, e_{m-n+1}\}$ and $e_i = u_i v_i$, $1 \leq i \leq m-n+1$. For any sequence $j_1, j_2, \cdots, j_{m-n+1}$, where $j_i \in \{0,1\}$, let $G(j_1, j_2, \cdots, j_{m-n+1})$ be the graph obtained from T^* by adding $m-n+1$ new vertices $w_1, w_2, \cdots, w_{m-n+1}$ and for every i with $1 \leq i \leq m-n+1$, adding a new edge joining w_i to v_i if $j_i = 0$ or joining w_i to u_i if $j_i = 1$. For example, $G(0, 0, \cdots, 0)$ is denoted by $T^* + v_1w_1 + v_2w_2 + \cdots + v_{m-n+1}w_{m-n+1}$. Then $G(j_1, j_2, \cdots, j_{m-n+1})$ is a tree with $m+1$ vertices and m edges. Let $H_{T^*}(G) = \{G(j_1, j_2, \cdots, j_{m-n+1}) \mid j_i \in \{0,1\}, 1 \leq i \leq m-n+1\}$. Thus $|H_{T^*}(G)| = 2^{m-n+1}$. If $m-n+1 = 0$ (i.e., $E(G) - E(T^*) = \phi$), then $G = T^*$ and $H_{T^*}(G) = \{G\}$.

Proposition 5. *(See [1].) Let G be a connected graph and T^* be a spanning tree of G. Then*

$$M\big(L(G)\big) = \sum_{T \in H_{T^*}(G)} M\big(L(T)\big).$$

Proposition 6. *(See [1].) Let G be a connected graph with n vertices and m edges, where m is even. Then $M(L(G)) \geq 2^{m-n+1}$, and the equality holds if $\Delta(G) \leq 3$, where $\Delta(G)$ is the maximum degree of G.*

Proposition 7. *(See [1].) Let G be a connected graph with n vertices and m edges, where m is even. Then $M(L(G)) = 2^{m-n+1}$ if and only if $f_G(u) + g_G(u) \in \{0, 2\}$ for each $u \in V(G)$.*

For any non-negative integer r and k, define

$$\eta(r) = \sum_{0 \leq k \leq r/2} \binom{r}{2k} (2k)!!.$$

Proposition 8. *(See [1].) Let G be a connected graph with n vertices and m edges, where m is even. Let x be any vertex in G such that $G-x$ is connected. Then*

$$M\big(L(G)\big) \geq \eta\big(d(x)\big) \cdot 2^{m-n-d(x)+2}.$$

3. The lower bound for the number of perfect matchings of line graphs

In this section, we will give a lower bound for $M(L(G))$ for any connected graph G with an even number of edges.

To prove the following, we define some notations. For two sets X and Y, let $X - Y = \{u \mid u \in X \text{ and } u \notin Y\}$. For $X \subseteq V(G)$ and $Y \subseteq V(G) - X$, let $E(X, Y) = \{xy \in E(G) \mid x \in X, y \in Y\}$. If $X = \{x\}$, denote $E(X, Y)$ by $E(x, Y)$. Let $e(x, Y) = |E(x, Y)|$.

Lemma 1. *Let G be a connected graph with n vertices and m edges, T^* be a spanning tree of G, and $u \in V(G)$ such that $G-u$ is connected. Then there exists a tree $T \in H_{T^*}(G)$ such that $p(T-u) = f_G(u) + g_G(u)$.*

Proof. Let $E(G) - E(T^*) = \{e_1, e_2, \cdots, e_{m-n+1}\}$ and $e_i = u_i v_i$, $1 \leq i \leq m-n+1$. Since $G-u$ is connected, $w(G-u) = 1$. We prove the result by induction on $f_G(u)$. If $f_G(u) = 0$, then $e(u, V(G-u)) = 1$, implying that $E_u \subseteq E(T^*)$. Hence $p(T-u) = g_T(u) = g_G(u) = f_G(u) + g_G(u)$ for each $T \in H_{T^*}(G)$. Assume that $f_G(u) = t$ $(t \geq 1)$ and the result holds when $0 \leq f_G(u) < t$.

Case 1. There exists an edge $e \in E_u - E(T^*)$.
Then $e \in \{e_1, e_2, \cdots, e_{m-n+1}\}$. Without loss of generality, we assume that $e = e_1$ and $u = u_1$. Since $e_1 \notin E(T^*)$, T^* is a spanning tree of $G - e_1$ and $G - e_1 - E(T^*) =$

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