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On packing arborescences in temporal networks

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ABSTRACT

A temporal network is a directed graph in which each arc has a time label specifying the time at which its end vertices communicate. An arborescence in a temporal network is said to be time-respecting, if the time labels on every directed path from the root in this arborescence are monotonically non-decreasing. In this paper, we consider a characterization of the existence of arc-disjoint time-respecting arborescences in temporal networks.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. For each directed graph D, we denote by $V(D)$ and $A(D)$ the sets of vertices and arcs of D, respectively. Furthermore, for each directed graph D and each vertex v of D, let $\delta_D(v)$ and $\varrho_D(v)$ be the sets of arcs of D leaving and entering v, respectively. We denote by $a = (u, v)$ an arc a from u to v.

A *temporal network* N is a pair (D, τ) of a directed graph D and a time label function $\tau\colon A(D) \to \mathbb{N}$. For each arc a of D, the time label $\tau(a)$ specifies the time at which its end vertices communicate. This model is used for modeling communication in distributed networks and scheduled transportation networks. See [1] for applications of temporal networks. If we communicate along a directed path P in a temporal network, then the time labels of the arcs of P must be monotonically non-decreasing. Formally speaking, a directed path P in a temporal network $N = (D, \tau)$ is said to be *time-respecting*, if

$$\tau(a_1) \le \tau(a_2) \le \cdots \le \tau(a_k),$$

where we assume that P passes through arcs $a_1, a_2, \ldots, a_k$ in this order.

Besides a directed path, an arborescences is another important concept in a directed graph from not only a theoretical point of view but also a practical point of view. Formally speaking, a subgraph T of a directed graph D with a specified vertex r is called an *r-arborescence* or an *arborescence rooted at r*, if

1. $V(T) = V(D)$,
2. there exists a directed path in T from r to every vertex v of D, and
3. for each vertex v of D,

$$|\varrho_T(v)| = \begin{cases} 0 & \text{if } v = r \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to see that an r-arborescence is a spanning tree in D whose arcs are directed away from r.

Assume that we are given a temporal network $N = (D, \tau)$ with a specified vertex r. For each r-arborescence

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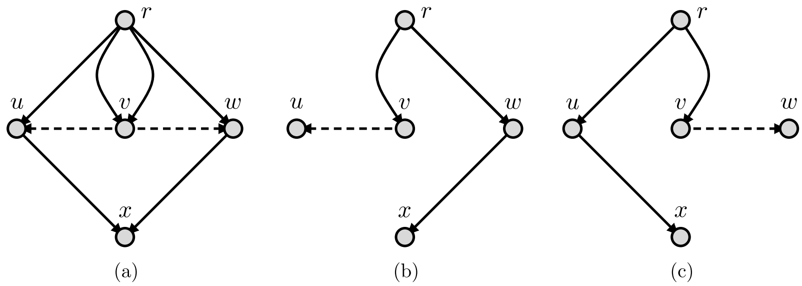

Fig. 1. (a) An example of a temporal network. A time label of each arc illustrated by a real line is equal to 1. A time label of each arc illustrated by a broken line is equal to 2. (b, c) Two arc-disjoint time-respecting r-arborescences in the temporal network illustrated in (a).

T in N and each vertex v of D with $v \neq r$, T is said to be *time-respecting on* v, if

$$\forall a \in \delta_T(v): \tau\big(\mathsf{in}(v)\big) \leq \tau(a), \tag{1}$$

where $\mathsf{in}(v)$ represents the unique arc in $\varrho_T(v)$. Furthermore, an r-arborescence T in N is said to be *time-respecting*, if T is time-respecting on every vertex v of D with $v \neq r$. It is not difficult to see that an r-arborescence T in N is time-respecting if and only if for every vertex v of T, the unique directed path from r to v in T is time-respecting.

In this paper, we consider a characterization of the existence of arc-disjoint time-respecting arborescences rooted at a specified vertex in a temporal network (see Fig. 1). Packing problems are very important problems in graph theory and combinatorial optimization. Furthermore, it is practically natural to think that a network in which there exist many arc-disjoint arborescences has high robustness against troubles.

2. Problem formulation

For defining our problem, we first consider the case where the time label of every arc is the same, i.e., we consider a characterization of the existence of arc-disjoint arborescences rooted at a specified vertex in a directed graph. For this case, the following important theorem was proved by Edmonds [2].

Theorem 1. *(See Edmonds [2].) For each directed graph D with a specified vertex r, there exist k arc-disjoint r-arborescences if and only if for every vertex v of D, there exist k arc-disjoint directed paths from r to v.*

Theorem 1 is one of the most important theorems in graph theory and combinatorial optimization. Furthermore, it gives us the following algorithmic implication. For checking the existence of k arc-disjoint r-arborescences, it is suffice to decide whether there exist k arc-disjoint directed paths from r to every vertex. Since we can decide in polynomial time whether there exist k arc-disjoint directed paths from r to every vertex (see, e.g., [3]), Theorem 1 implies that we can decide in polynomial time whether there exist k arc-disjoint r-arborescences. It should be noted that Theorem 1 was extended to various settings (see, e.g., [4–6]).

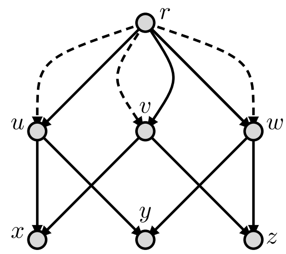

Fig. 2. A counterexample. A time label of each arc illustrated by a real line is equal to 1. A time label of each arc illustrated by a broken line is equal to 2.

In this paper, we consider the following extension of Theorem 1. "For each temporal network $N = (D, \tau)$ with a specified vertex r, there exist k arc-disjoint time-respecting r-arborescences if and only if for every vertex v of D, there exist k arc-disjoint time-respecting directed paths from r to v." Unfortunately, it is known [1] that this statement is not always true. Although the counterexample given in [1] has a directed cycle, we can construct an acyclic temporal network in which this statement is not true by slightly modifying the counterexample in [1] (see Fig. 2), where a temporal network $N = (D, \tau)$ is said to be *acyclic*, if D is acyclic.

In this paper, we consider the above statement in some special case. Precisely speaking, we consider the case where an input temporal network satisfies the pre-flow condition. See the next section for the definition of the pre-flow condition. The pre-flow condition in a directed graph was introduced in [7]. Our definition is a natural extension of this. It should be noted that the temporal network illustrated in Fig. 2 is not pre-flow.

3. Our contributions

Assume that we are given a temporal network $N = (D, \tau)$ with a specified vertex r. A subgraph T of D is called a *partial r-arborescence*, if $r \in V(T)$ and T is an r-arborescence in the subgraph of D induced by $V(T)$. Notice that $V(T)$ is not necessarily equal to $V(D)$. For each vertex v of D with $v \neq r$, we define $\lambda_N(v)$ as the maximum number of arc-disjoint time-respecting directed paths from r to v in N. In addition, define $\lambda_N(r) := \infty$. A partial r-arborescence in N is said to be *time-respecting*, if (1) holds for every vertex v of T with $v \neq r$. For each