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The clique-transversal set problem in claw-free graphs with degree at most 4

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ABSTRACT

A clique of a graph G is defined as a complete subgraph maximal under inclusion and having at least two vertices. A clique-transversal set D of G is a subset of vertices of G such that D meets all cliques of G. The clique-transversal set problem is to find a minimum clique-transversal set of G. In this paper we present a polynomial time algorithm for the clique-transversal set problem on claw-free graphs with degree at most 4.

1. Introduction

All graphs considered here are finite, simple and nonempty. For standard terminology not given here we refer the reader to [5]. Let $G = (V, E)$ be a graph with *vertex set* V and *edge set* E. The number of vertices of G is called the *order* of G. For a vertex $v \in V$, the *open neighborhood* $N(v)$ of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u : uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Every vertex in $N(v)$ is also called a *neighbor* of v. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. By $\delta(G)$ and $\Delta(G)$, we denote the *minimum degree* and the *maximum degree* of graph G, respectively. A graph G is said to be *k-regular* if $d_G(v) = k$ for all $v \in V$. In particular, a 3-regular graph is also called a *cubic* graph. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality m and n; K_n is the complete graph on n vertices. The graph $K_{1,3}$ is also called a *claw*, and K_3 a *triangle*. The graph $K_4 - e$ (obtained from K_4 by deleting one edge) is called a *diamond*. For a subset $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. For a given graph F, we say that a graph G is F-free if it does not contain F as an induced subgraph. In particular, a $K_{1,3}$-free graph is *claw-free*. For a family of graphs $\{F_1, \dots, F_k\}$, we say that G is $\{F_1, \dots, F_k\}$-free if it is F_i-free for all $i = 1, \dots, k$. A *matching* in a graph G is a set of pairwise nonadjacent edges. The *matching number*, denoted by $\alpha'(G)$, is the cardinality of a maximum matching of G. An *edge covering* of G is a set of edges that meets every vertex of G. The number of edges in a minimum edge covering of G without isolated vertices is denoted by $\beta'(G)$. For any graph of order n without isolated vertices, it is easy to verify that $\alpha'(G) + \beta'(G) = n$.

A *clique* C of a graph G is a complete subgraph maximal under inclusion and having at least two vertices. According to this definition, isolated vertices are not considered to be cliques here. A clique of order m of G is called an *m-clique* of G. A set $D \subseteq V$ is called a *clique-transversal set* of G, if

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D meets all cliques of G, i.e., $D \cap V(C) \neq \emptyset$ for any clique C of G. The *clique-transversal number*, denoted by $\tau_C(G)$, is the minimum cardinality of a clique-transversal set of G. The *clique-transversal set problem* (abbreviated to CTS) is to find a minimum clique-transversal set of G.

To motivate the study of the clique-transversal sets in graphs, we present an application in terms of communication networks. Consider a graph associated with a communication network where the vertices in the graph correspond to the sites of the network. A clique usually represents a cluster of sites that have the best possible ability to rapidly exchange information among the members of the cluster. The clique-transversal set in the graph is faster to control all clusters and keeps the ability of dominating the whole network.

Erdős, Gallai and Tuza [12] observed the clique-transversal set problem for an arbitrary graph is NP-hard. Chang, Farber and Tuza [7] proved the clique-transversal set problem is NP-hard on split graphs. Guruswami and Rangan [13] proved that the clique-transversal set problem is still NP-hard on cocomparability, planar, line and total graphs. However, there are polynomial time algorithms to find the minimum clique-transversal set for comparability graphs [4], strongly chordal graphs [7,8,13], chordal graphs with bounded clique size [13], k-trees with bounded k [8], balanced graphs [6], short-chorded graphs with no 3-fans nor 4-wheels [11], distance-hereditaty graphs [16], Helly circular-arc graphs [13] and $\overline{3K_2}$-free circular-arc graphs [10]. The bounds of clique-transversal number of graphs were extensively studied in [1–3,9,12,19] and elsewhere.

We proved that the clique-transversal set problem is NP-complete on cubic graphs [15]. In [13] the authors proved that the clique-transversal set problem is NP-hard on line graphs with maximum degree 12. This implies that the problem is also NP-hard on claw-free graphs with maximum degree 12. In this paper we give a polynomial time algorithm for the clique-transversal set problem in claw-free graphs with degree at most 4.

2. Preliminaries

Let us introduce some more notation and terminology. By starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, one obtains the *join* of G and H, denoted by $G \vee H$. The join $C_n \vee K_1$ of a cycle C_n and a single vertex is referred to as a *n-wheel* with *n spokes* and denoted by W_n. Let us call a graph of the form $C_n \vee \overline{K_2}$ $(n \geq 4)$ a *double n-wheel*. The join $P_n \vee K_1$ of a path P_n and a single vertex is referred to as a *n-fan* with *n spokes* and denoted by F_n. A graph is called a *domino* if every vertex is contained in at most two cliques. Dominos are a sub-class of claw-free graphs and were studied by Kloks et al. [14]. Let $R(G)$ be the graph obtained from G by identifying all the vertices with the same closed neighborhood.

For dominos, the following lemma due to Kloks et al. [14].

Lemma 1. *(See [14].) If G is a domino, then we have the following statements.*

(1) *There exists a linear time algorithm to enumerate all the cliques of G.*
(2) *The graph R(G) can obtain from G in linear time.*
(3) *A graph G is a domino if and only if R(G) is a {claw, diamond}-free graph.*
(4) *A graph G is a domino if and only if G is a {4-wheel, 4-fan, claw}-free graph.*

Lemma 2. *If G is a {claw, diamond}-free graph, then there exists a polynomial time algorithm for the clique-transversal set problem of G.*

Proof. Without loss of generality, we may assume that G is connected and G is not complete. If G is a {claw, diamond}-free graph, then we can easily check that G is the line graph of a triangle-free graph H, otherwise there would be a diamond in G. Graph H can be obtained in linear time by the algorithm of Roussopoulos (see [18]). So, the clique-transversal set problem of G is equivalent to find a minimum edge cover in the graph obtained from H by deleting pendant vertices. Since this problem is polynomially computed in $O(\sqrt{|V(H)|}|E(H)|)$ by finding a maximum matching and extending it greedily so that all vertices are covered [17], the clique-transversal set problem of G can be solved in polynomial time. □

Theorem 3. *There is a polynomial time algorithm for the clique-transversal set problem in a domino graph G.*

Proof. By Lemma 1, we see that the graph $R(G)$ can be obtained from G in linear time, and $R(G)$ is {claw, diamond}-free. It follows from Lemma 2 that the clique-transversal set problem in $R(G)$ can be calculated in polynomial time. But clearly $\tau_C(G) = \tau_C(R(G))$ by the construction of $R(G)$, and hence the clique-transversal set problem of G can be calculated in polynomial time. □

Note that each vertex in a claw-free graph with $\Delta(G) \leq 3$ lies in at most two cliques. By Theorem 3, we immediately get the following result.

Corollary 4. *If G is a claw-free graph with $\Delta(G) \leq 3$, then there exists a polynomial time algorithm for the clique-transversal set problem.*

3. A polynomial time algorithm for CTS problem on claw-free graph with maximum degree 4

In this section, we present a polynomial time algorithm for clique-transversal set problem in a claw-free graph G with $\Delta(G) = 4$. For a claw-free graph G with $\Delta(G) = 4$, we have the following result by observing that a vertex of G is possibly contained in three cliques.

Lemma 5. *For a claw-free graph G with $\Delta(G) \leq 4$, let $M(G)$ denote the set of vertices of G each of which lies in at least three cliques of G. If $|M(G)| \geq 1$, then we can construct a claw-free graph G^* with $\Delta(G^*) \leq 4$ and $|M(G^*)| < |M(G)|$, and a minimum clique-transversal set of G can be obtained from a minimum clique-transversal set of G^*.*