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The transportation metric and related problems

Yaron Ostrovsky-Berman

School of Engineering and Computer Science, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

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1. Introduction and related work

Public transportation networks provide time saving routes with discrete entry and exit points. The network, described as an undirected graph with positive edge weights proportional to travel time, induces a metric on the plane. In this paper we describe the properties of this metric, give the first optimal time algorithm for the construction of shortest path maps and Voronoi diagrams in this metric, and discuss practical issues in its implementation.

There have been many attempts to model the complexity of real world terrain with a simple mathematical model. The most general of these is the weighted region model [8] in which the plane is divided into regions with weights corresponding to the difficulty of crossing the terrain. There is no efficient algorithm for finding shortest paths in this general model without approximating the solution. One important special case is shortest paths amidst polygonal obstacles in the plane, in which the obstacles have infinite weight and the free space has unit weight. Mitchell [6] gave the first sub-quadratic solution, which runs in $O(n^{3/2+\varepsilon})$ time. His result was improved to optimal $O(n \log n)$ time by Hershberger and Suri [5]. Both use the continuous Dijkstra paradigm [7]. Recently, two models for urban environments were introduced: a rectilinear transportation network in the L_1 metric [2], and a single highway passing through a city [1]. The Voronoi diagram in both models can be constructed in optimal $O(n \log n)$ time.

The transportation network proposed in this paper cannot be modeled with weighted regions because it allows crossing the transportation line at no time cost, but restricts entrance to and exit from the network to the fixed stations. The metric induced by the transportation network was briefly introduced by Aurenhammer and Klein [3]. The first algorithm for the Voronoi diagram in this metric (called the airlift Voronoi diagram) was given by Aichholzer et al. [2] with $O(k^2 + (n+k)\log(n+k))$ time complexity, where n is the number of sites and k is the number of stations. The authors also posed the question of whether there exists a matching lower bound for

E-mail address: yaronber@cs.huji.ac.il (Y. Ostrovsky-Berman).

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the time complexity. Their result was improved by Ostrovsky-Berman [9] with an input sensitive algorithm having the same worst case complexity, but provably better bounds under realistic regularity assumptions on the input. Both these methods solve the problem by computing appropriate weights to the stations and constructing the additively weighted Voronoi Diagram (AWVD) of the sites and stations. In this paper, we settle the question of the time complexity by combining the continuous Dijkstra method with the reduction to AWVD to get optimal $O(k \log k + e)$ and $O((n+k)\log(n+k)+e)$ time algorithms for the shortest path map and Voronoi diagram, respectively, where e denotes the number of transportation lines in the network.

2. Definition and properties

Let $T \subset \mathbb{R}^2$ denote a set of station positions in the plane, and let E denote the connection relation between the stations, such that $(t_1, t_2) \in E$ if and only if $t_1, t_2 \in T$ and the stations are connected by a line. Denote the positive weight of a transportation line $(t_1, t_2) \in E$ by $w(t_1, t_2)$ and define it to be infinite when the points are not connected stations. The graph $\langle T, E, w\rangle$ describes the transportation network. For the Voronoi diagram problem, the set $S \subset \mathbb{R}^2$ denotes the sites. Let $d(p, q)$ denote the Euclidean distance between two points in the plane. The transportation distance between two points is defined recursively as follows:

$$d_T(p,q) = \min\Big\{d(p,q), w(p,q), \min_{t\in T}\big\{d_T(p,t) + d_T(t,q)\big\}\Big\}.$$

The unit disc of a point p in the metric induced by the transportation distance,

$$B_T(p) = \big\{x \mid d_T(p,x) \leqslant 1\big\},$$

depends on the proximity of p to the network (Fig. 1). This leads to the following properties:

(1) The Voronoi cell of a site can have several connected components;
(2) The transportation distance bisector of two points p and q, $b_T(p,q) = \{x \mid d_T(p,x) = d_T(q,x)\}$, consists of line segments (points whose shortest path to p and q is a straight line), hyperbolic arcs (points whose shortest path to p or q uses the transportation network), and star shaped regions bounded by line and hyperbolic segments (points whose shortest paths to p and q use at least one shared network station).

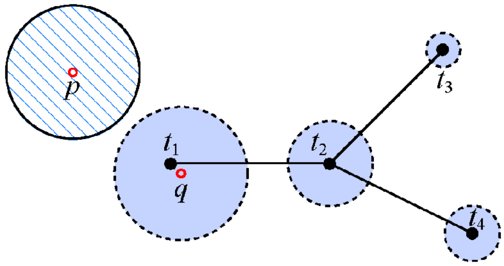

Fig. 1. The unit disc in the transportation metric. Solid lines are transportation lines. The hatched disc is the unit disc of p, and the shaded discs comprise the unit disc of q.

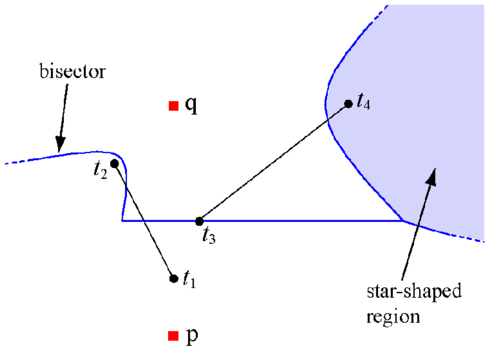

Fig. 2. The bisector of two points. Solid circles are stations, the thin solid lines connecting them are transportation lines. The weight of the transportation lines is tenth of the Euclidean distance between the stations. The thick curve is the bisector of p and q. The shaded region is part of the bisector because $d_T(p,t_4) = d_T(q,t_4)$.

Fig. 2 illustrates these cases.

As shown in [9], the shortest path map is constructed by computing the transportation distance from the source to all the stations, then using this distance as a negative weight in the AWVD of the source and the stations. In the AWVD, sites are treated as discs with radii equal to their weights. Thus the distance between a point p and a site s with weight w_s is $d_{\mathrm{AWVD}}(p,s) = d(p,s) - w_s$. Fortune [4] showed how to construct the AWVD in $O(n \log n)$ time. The transportation Voronoi diagram (TVD) is constructed similarly, only now the weights are assigned according to the *closest* site in transportation distance.