Journal of Computer and System Sciences 80 (2014) 837–848

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

On the connectivity preserving minimum cut problem

Qi Duan [a,*], Jinhui Xu [b,1]

[a] Department of Software and Information Systems, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[b] Department of Computer Science and Engineering, State University of New York at Buffalo, Buffalo, NY 14260, USA

ARTICLE INFO

Article history:
Received 14 February 2012
Received in revised form 26 November 2013
Accepted 8 January 2014
Available online 15 January 2014

Keywords:
Minimum cut
Inapproximability
Connectivity preserving

ABSTRACT

In this paper, we study a generalization of the classical minimum cut problem, called *Connectivity Preserving Minimum Cut* (*CPMC*) problem, which seeks a minimum cut to separate a pair (or pairs) of source and destination nodes and meanwhile ensure the connectivity between the source and its partner node(s). For this problem, we consider two variants, connectivity preserving minimum node cut (CPMNC) and connectivity preserving minimum edge cut (CPMEC). For CPMNC, we show that it cannot be approximated within $\alpha \log n$ for some constant α unless $P = NP$, and cannot be approximated within any $poly(\log n)$ unless NP has quasi-polynomial time algorithms. The hardness results hold even for graphs with unit weight and bipartite graphs. For CPMEC, we show that it is in P for planar graphs.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

Minimum cut is one of the most fundamental problems in computer science and has numerous applications in many different areas [1–4]. In this paper, we consider a new generalization of the minimum cut problem, called *connectivity preserving minimum cut* (*CPMC*) problem arising in several areas. In this problem, we are given a connected graph $G = (V, E)$ with positive node (or edge) weights, a source node s_1 and its partner node s_2, and a destination node t. The objective is to compute a cut with minimum weight to disconnect the source s_1 and destination t, and meanwhile preserve the connectivity of s_1 and its partner node s_2 (i.e., s_1 and s_2 are connected after the cut). The weights can be associated with either the nodes (i.e., vertices) or the edges, and accordingly the cut can be either a set of nodes, called a connectivity preserving node cut, or a set of edges, called a connectivity preserving edge cut. Corresponding to the two types of cuts, the CPMC problem has two variants, *connectivity preserving minimum node cut* (*CPMNC*) and *connectivity preserving minimum edge cut* (*CPMEC*).

The CPMC problem has both theoretical and practical importance. Theoretically, it is closely related to three fundamental problems, minimum cut, set cover, and shortest path. Practically, the CPMC problem finds applications in many different areas. In network security, for example, CPMC can be used to identify potential nodes for attacking. In such applications, an attacker (or police) may want to intercept all communication (or traffic) between a source node s_1 and a destination node t. It is possible that some nodes with (direct) connection to the destination might already have been compromised. To maximally utilize such nodes, the attacker only needs to compromise another set of nodes with minimum cost so that all traffic between the source and destination nodes passes one of the compromised nodes. To solve this problem, one can formulate it as a CPMC problem in which the compromised nodes are treated as partners of the source after removing their connections to the destination. In applications related to network reliability and emergency recovery, a node in a network

* Corresponding author.
E-mail addresses: qduan@uncc.edu (Q. Duan), jinhui@buffalo.edu (J. Xu).
[1] The research of the author was supported in part by NSF under grant IIS-1115220.

0022-0000/$ – see front matter © 2014 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcss.2014.01.003

might be contaminated, and has to be separated from some critical nodes. Meanwhile, traffic flows among the critical nodes have to be maintained with a minimum cost. To solve such a problem, one can treat the critical nodes as the source and partner nodes and the contaminated node as the destination node, and formulates it as a CPMC problem. In data mining, machine learning, and image segmentation, CPMC can be used to model clustering or segmentation problems with additional constrains for clustering or segmenting certain objects together.

The CPMC problem can be generalized in several ways. For example, we may have multiple pairs of source and destination nodes, and each source node may have multiple partner nodes. The simplest version is the 3-node case in which only one source node, one destination node, and one partner node exist. Note that the 3-node case is much different from the minimum 3-terminal cut problem [5] in which all three nodes are required to be separated, whereas in the 3-node case two nodes are required to be connected. In this paper, we will mainly focus on the 3-node case.

The CPMC problem is in general quite challenging, even for the 3-node case. One of the main reasons is that the connectivity preserving requirement and the minimum cut requirement seem to be contradicting to each other. As it will be shown later, the hardness of the CPMC problem increases dramatically with the added connectivity requirement. This phenomenon (i.e., increased hardness with the additional connectivity constraint) is consistent with the observations by Yannakakis [6] in several other graph related optimization problems.

The CPMC problem is a new and interesting problem. To the best of our knowledge, it has not been studied previously. Related problems include the non-separating cycle and optimal cycle problems in certain surfaces [7,8]. Since there is no restriction on the source and its partner nodes, CPMC seems to be more general and fundamental.

In this paper, we mainly consider CPMNC, CPMEC, and CPMC in planar graphs. For the CPMNC problem, we show that the problem is extremely hard to solve and to approximate, even for some very special cases. Particularly, we show that it cannot be approximated within a factor of $\alpha\, logn$ for some small constant α unless $P = NP$. We also use Feige and Lovasz's two-prover one round interactive proof protocol [9] to show that the CPMNC problem cannot be approximated within any $poly(logn)$ factor unless $NP \subset DTIME(n^{poly(logn)})$. The hardness results hold even for unit-weighted graphs and bipartite graphs.

For planar graphs, we show that the CPMNC problem can be solved in polynomial time if s_1 and s_2 are on the same face. For the CPMEC problem, we present a polynomial time solution for general planar graphs, which can be used for CPMC applications in image processing and machine learning. We also reveal a close relation between a Location Constrained Shortest Path (LCSP) problem and the CPMEC problem in special planar graphs in which s_1 and t are in the same face, and give polynomial time solutions to both problems.

2. Connectivity preserving node cut problem

First we note that the CPMNC problem is an NP optimization problem. To determine whether a valid cut exists, one just needs to check if t is connected to any bridge node between s_1 and s_2; if so, then no valid cut exists. Clearly, this can be done in polynomial time. Thus, we assume thereafter that a cut always exists.

We first define the decision version of the CPMNC problem.

Definition 2.1 *(Decision problem of CPMNC).* Given an undirected graph $G = (V, E)$ with each node $v_i \in V$ associated with a positive integer weight c_i, a source node s_1, a partner node s_2, a destination node t, and an integer $b > 0$, determine whether there exists a subset of nodes in V with total weight less than or equal to b such that the removal of this subset disconnects t from s_1 but preserves the connectivity between s_1 and s_2.

The decision version of the CPMEC problem can be defined similarly.

Theorem 2.2. *The CPMNC problem is NP-complete and cannot be approximated within α_1 logn for some constant α_1 unless $P = NP$, where $n = |V|$.*

Proof. To prove the theorem, we reduce the set cover problem to this problem. In the set cover problem, we have a ground set $\mathcal{T} = \{e_1, e_2, \ldots, e_{n_1}\}$ of n_1 elements, and a set $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of k subsets of $\mathcal{T}$ with each $S_i \in \mathcal{S}$ associated with a weight w_i. The objective is to select a set $\mathcal{O}$ of subsets in $\mathcal{S}$ so that the union of all subsets in $\mathcal{O}$ contains every element in $\mathcal{T}$ and the total weight of subsets in $\mathcal{O}$ is minimized.

Given an instance I of the set cover problem with n_1 elements and k sets, we construct a new graph. The new graph has an element gadget for every element, and every element gadget contains $k_1 + 2$ nodes, where k_1 is the number of sets that contains this element. In every gadget, there are two end points, and k_1 internal nodes are connected to the two end nodes in parallel. Every internal node of a gadget corresponds to a set that contains this element. All such n_1 gadgets are connected sequentially through their end points, with s_1 and s_2 at the two ends of the whole construction. All nodes correspond to the same set are connected to a new node which we call set node, and all set nodes are connected to t. Fig. 1 is the graph constructed for set cover instance with three elements x_1, x_2, and x_3, three sets $A_1 = \{x_1, x_3\}$, $A_2 = \{x_2, x_3\}$, and $A_3 = \{x_1, x_2\}$.

Every set node is assigned a weight $w_i n_1 k$, where w_i is the weight of the corresponding set in the original set cover instance. All other nodes are assigned weight 1. We let $b = n_1 k D_1 + n_1 k - 1$, where D_1 is the upper bound of weight in

Download English Version:

https://daneshyari.com/en/article/10332785

Download Persian Version:

https://daneshyari.com/article/10332785

Daneshyari.com