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On some network design problems with degree constraints ☆

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ABSTRACT

We study several network design problems with degree constraints. For the minimum-cost Degree-Constrained 2-Node-Connected Subgraph problem, we obtain the first non-trivial bicriteria approximation algorithm, with $5b(v)+3$ violation for the degrees and a 4-approximation for the cost. This improves upon the logarithmic degree violation and no cost guarantee obtained by Feder, Motwani, and Zhu (2006). Then we consider the problem of finding an arborescence with at least k terminals and with minimum maximum outdegree. We show that the natural LP-relaxation has a gap of $\Omega(\sqrt{k})$ or $\Omega(n^{1/4})$ with respect to the multiplicative degree bound violation. We overcome this hurdle by a combinatorial $O(\sqrt{(k\log k)/\Delta^*})$-approximation algorithm, where Δ^* denotes the maximum degree in the optimum solution. We also give an $\Omega(\log n)$ lower bound on approximating this problem. Then we consider the undirected version of this problem, however, with an extra diameter constraint, and give an $\Omega(\log n)$ lower bound on the approximability of this version. We also consider a closely related Prize-Collecting Degree-Constrained Edge-Connectivity Survivable Network problem, and obtain several results in this direction by reducing the prize-collecting variant to the regular one. Finally, we consider the Terminal Steiner Tree problem, which is a simple variant of the Degree-Constrained Steiner Tree problem, when some terminals are required to be leaves. We show that this seemingly simple problem is equivalent to the Group Steiner Tree problem.

1. Introduction

1.1. Problems considered

In network design problems one seeks a cheap subgraph H of a given graph G that satisfies some given properties. In the b-Matching problem H should satisfy prescribed degree constraints, while in the Survivable Network problem H should satisfy prescribed connectivity requirements. The Degree-Constrained Survivable Network problems are a combination of these two fundamental problems, where H should satisfy both degree constraints and connectivity requirements. For most of these problems, even checking whether there exists a feasible solution is NP-hard, hence one considers a bicriteria approximation when the degree constraints are relaxed. Namely, the goal is to compute a cheap solution that satisfies the connectivity requirements and has small degree violation.

☆ A preliminary version of this paper is Khandekar et al. (2011) [16].

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Many recent papers considered *edge-connectivity* Degree-Constrained Survivable Network problems, see a recent survey in [20]. Our first problem is the simplest *node-connectivity* problem. A graph H is k-(node-)connected if it contains k internally disjoint paths between every pair of its nodes. In the k-Connected Subgraph problem we are given a graph $G=(V,E)$ with edge-costs and an integer k. The goal is to find a minimum-cost k-connected spanning subgraph H of G. In the Degree-Constrained k-Connected Subgraph problem, we are also given degree bounds $\{b(v)\colon v \in B \subseteq V\}$. The goal is to find a minimum-cost k-connected spanning subgraph H of G such that in H, the degree of every node $v \in B$ is at most $b(v)$. We consider the case $k=2$.

> Degree-Constrained 2-Connected Subgraph
> *Instance*: An undirected graph $G=(V,E)$ with non-negative edge-costs $\{c_e\colon e \in E\}$, and degree bounds $\{b(v)\colon v \in B \subseteq V\}$.
> *Objective*: Find a minimum cost 2-connected spanning subgraph H of G that satisfies the *degree constraints* $\deg_H(v) \leqslant b(v)$ for all $v \in B$.

In the Steiner k-Tree problem one seeks a minimum-cost tree that contains at least k-terminals (when every node is a terminal we get the k-MST problem). Our next problem is the minimum-degree directed version of this problem. Given a directed graph G, a set S of terminals, and an integer $k \leqslant |S|$, a *k-arborescence* is an arborescence in G that contains k terminals; in the case of undirected graphs we have a *k-tree*. For a directed/undirected graph or edge-set H let $\Delta(H)$ denote the maximum outdegree/degree of a node in H.

> Minimum Degree k-Arborescence
> *Instance*: A directed graph $G=(V,E)$, a root $s \in V$, a subset $S \subseteq V \setminus \{s\}$ of terminals, and an integer $k \leqslant |S|$.
> *Objective*: Find in G a k-arborescence T rooted at s that minimizes $\Delta(T)$.

The origin of this problem is in peer-to-peer networking, when one wants to bound the maximum load (degree) of a node, while connecting the root to the maximum number of terminals. It is also of interest to bound the height of such a tree, to limit the time for sending messages from the root. This motivates our next problem, for which we only show a lower bound. Hence we show it for the *less* general case of undirected graphs.

> Degree and Diameter Bounded k-Tree
> *Instance*: An undirected graph $G=(V,E)$, a subset $S \subseteq V$ of terminals, an integer $k \leqslant |S|$, and a diameter bound D.
> *Objective*: Find a k-tree T with diameter bounded by D that minimizes $\Delta(T)$.

Let $\lambda_H(u,v)$ denote the maximum number of edge-disjoint uv-paths in H. In the Edge-Connectivity Survivable Network problem we are given a graph $G=(V,E)$ with edge-costs, a collection $\mathcal{P}=\{\{u_1,v_1\},\ldots,\{u_k,v_k\}\}$ of node pairs, and connectivity requirements $\mathcal{R}=\{r_1,\ldots,r_k\}$. The goal is to find a minimum-cost subgraph H of G that satisfies the connectivity requirements $\lambda_H(u_i,v_i) \geqslant r_i$ for all i.

We consider a combination of the following two generalizations of this problem. In Degree-Constrained Edge-Connectivity Survivable Network, we are given degree bounds $\{b(v)\colon v \in B\}$. The goal is to find a minimum-cost subgraph H of G that satisfies the connectivity requirements and the degree constraints $\deg_H(v) \leqslant b(v)$ for all $v \in B$. In the Prize-Collecting Edge-Connectivity Survivable Network we are given a submodular monotone non-decreasing penalty function $\pi : 2^{\{1,\ldots,k\}} \to \mathbb{R}_+$ (π is given by an evaluation oracle). The goal is to find a subgraph H of G that minimizes the *value* $\mathtt{val}(H)=c(H)+\pi(\mathtt{unsat}(H))$ of H, where $\mathtt{unsat}(H)=\{i \mid \lambda_H^S(u_i,v_i)<r_i\}$ is the set of requirements *not* (completely) satisfied by H. Formally, the problem we consider is as follows.

> Prize-Collecting Degree-Constrained Edge-Connectivity Survivable Network
> *Instance*: An undirected graph $G=(V,E)$ with non-negative edge-costs $\{c_e\colon e \in E\}$, a collection $\mathcal{P}=\{\{u_1,v_1\},\ldots,\{u_k,v_k\}\}$ of node pairs, connectivity requirements $\mathcal{R}=\{r_1,\ldots,r_k\}$, a submodular monotone non-decreasing penalty function $\pi : 2^{\{1,\ldots,k\}} \to \mathbb{R}_+$ given by an evaluation oracle, and degree bounds $\{b(v)\colon v \in B \subseteq V\}$.
> *Objective*: Find a subgraph H of G that satisfies the *degree constraints* $\deg_H(v) \leqslant b(v)$ for all $v \in B$, and minimizes the *value*
>
> $$\mathtt{val}(H)=c(H)+\pi\big(\mathtt{unsat}(H)\big)$$
>
> of H, where $\mathtt{unsat}(H)=\{i \mid \lambda_H^S(u_i,v_i)<r_i\}$ is the set of requirements *not* satisfied by H.

The Steiner Tree problem is a particular case of this problem, when we seek a minimum-cost subtree T of G that contains a specified subset S of terminals. In the degree constrained version of Steiner Tree, we are also given degree bounds on nodes