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A bi-intuitionistic modal logic: Foundations and automation

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ABSTRACT

The paper introduces a bi-intuitionistic modal logic, called BISKT, with two adjoint pairs of tense operators. The semantics of BISKT is defined using Kripke models in which the set of worlds carries a pre-order relation as well as an accessibility relation, and the two relations are linked by a stability condition. A special case of these models arises from graphs in which the worlds are interpreted as nodes and edges of graphs, and formulae represent subgraphs. The pre-order is the incidence structure of the graphs. We present a comprehensive study of the logic, giving decidability, complexity and correspondence results. We also show the logic has the effective finite model property. We present a sound, complete and terminating tableau calculus for the logic and use the MetTeL system to explore implementations of different versions of the calculus. An experimental evaluation gave good results for satisfiable problems using predecessor blocking.

1. Introduction

The paper introduces a bi-intuitionistic modal logic, called BISKT, with two adjoint pairs of tense operators. A special case of the semantics for this logic has Kripke models in which the worlds are interpreted as nodes and edges of graphs, and formulae represent subgraphs. In this context the accessibility relations are essentially 'relations on graphs'. To motivate the logic and the structures which underlie its semantics, we start by reviewing the motivation for developing a theory of 'relations on graphs', which generalizes that of 'relations on sets'. One novel feature of relations on graphs is a pair of adjoint converse operations instead of the involution found with relations on sets. One half of this pair (the 'left converse') plays an essential role later in defining the relational semantics of BISKT.

Relations on sets underlie the most fundamental of the operations used in the body of techniques for image processing known as mathematical morphology [23,9]. Using $\mathbb{Z}^2$ to model a grid of pixels, binary (i.e., black and white) images are modelled by subsets of $\mathbb{Z}^2$. One aim of processing images is to accentuate significant features and to lessen the visual impact of the less important aspects. Several basic transformations on images are parameterized by small patterns of pixels called structuring elements. These structuring elements generate relations, which transform subsets of $\mathbb{Z}^2$ via the correspondence between relations $R \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$ and union-preserving operations on the powerset $\mathscr{P}(\mathbb{Z}^2)$. Several fundamental properties of image processing operations can be derived using only properties of these relations.

There have been various proposals for developing a version of mathematical morphology for graphs, one of the earliest being Heijmans and Vincent's [12]. One direction in more recent work is seen in [10]. However, most work in this area does not use a relational approach, probably because a theory of relations on graphs may be constructed in several dif-

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ferent ways. In one way [24,25], the set of relations on a graph, or more generally a hypergraph, forms a generalization of a relation algebra where, in particular, the usual involutive converse operation becomes a pair of operations (the left converse and the right converse) forming an adjoint pair. In the present paper, we use these relations to give a semantics for a bi-intuitionistic modal logic in which propositions are interpreted over subgraphs as opposed to subsets of worlds as in standard Kripke semantics.

Accessibility relations with additional structure are already well-known in intuitionistic modal logic [29]. However, the semantics for the logic we present is distinguished both from this work, and from existing related work we discuss in Section 8, by the use of the left-converse operation. This means the logic BISKT has novel features which include $\Diamond\phi$ being equivalent to $\neg\!\!\!\!\!\!\;\;\Box\neg\phi$, where $⌐$ and $\neg$ are the two negations present in BISKT.

Connections between mathematical morphology and modal logic have been developed by Aiello and Ottens [1], who implemented a hybrid modal logic for spatial reasoning, and by Aiello and van Benthem [2] who pointed out connections with linear logic. Bloch [8], also motivated by applications to spatial reasoning, exploited connections between relational semantics for modal logic and mathematical morphology. These approaches used morphology operations on sets, and one motivation for our own work is to extend these techniques to relations on graphs.

In this paper, we focus on the logic itself, and the semantic setting we use is more general than that arising from relations on graphs or hypergraphs. We justify the applicability of the logic by explaining its role in developing the theory of mathematical morphology for graphs as a generalization of the set-based case.

The main contribution of the paper is a *bi-intuitionistic tense logic*, called BISKT, for which a Kripke frame consists of a pre-order H interacting with an accessibility relation R via a *stability* condition.[1] The semantics interprets formulae as H-sets, the downward-closed subsets of the pre-order. A particular case arises when the worlds represent the edges and nodes of a graph and formulae are interpreted as subgraphs. We establish that BISKT is decidable and has the effective finite model property.

The semantic setting for BISKT is a relational setting in which deduction calculi can be developed systematically. We follow the methodology of tableau calculus synthesis and refinement as introduced in [21,26] to develop a tableau calculus for the logic. We give soundness, completeness and termination results as consequences of results of tableau synthesis and the effective finite model property of BISKT.

Implementing a prover is normally a time-consuming undertaking but MetTeL[2] is software for automatically generating a tableau prover from a set of tableau rules given by the user [27]. For us, using MetTeL turned out to be useful because we could experiment with implementations of different versions of the calculus. In combination with the tableau synthesis method, it was easy to run tests on a growing collection of problems with different provers for several preliminary versions of formalizations of bi-intuitionistic tense logics before settling on the definition given in this paper. MetTeL has also allowed to us experiment with different refinements of the rules and different forms of blocking. Blocking is a technique for forcing termination of tableau calculi for decidable logics.

The paper is structured as follows. Section 2 presents the basic notions of pre-orders and relations on downward-closed sets as well as on graphs. Relations on graphs occur naturally in mathematical morphology as demonstrated by an example in Section 2.5. Section 3 defines BISKT as a bi-intuitionistic stable tense logic with a semantics in which formulae are interpreted as downward-closed sets, which include subgraphs as a special case. Just as intuitionistic propositional logic can be embedded into the modal logic *S4*, BISKT can be embedded into a multi-modal logic with *S4* modalities, which allows us to give decidability, complexity and correspondence results for BISKT. In Section 4, the applicability of the logic to mathematical morphology on graphs is considered. The need for automated deduction tools for BISKT in the exploration of the properties of morphology operations on graphs is explained. A systematic way of deriving correspondence results for BISKT is presented in Section 5 along with a demonstration that the frame condition corresponding to the Löb formula in the classical case also holds for BISKT. Frame conditions can be seen as properties of generalized structuring elements, so Section 5 is also a necessary foundation for the programme of using BISKT as a tool to develop mathematical morphology on graphs. In Section 6, we present a terminating labelled tableau calculus for BISKT. Results and findings of an evaluation of different presentations of the tableau calculus implemented using the tableau prover generator MetTeL are given in Section 7. Connections to other work are discussed in Section 8.

2. Relations on pre-orders and on graphs

2.1. The algebraic structure of H-sets

Let U be a set with a subset $X \subseteq U$, and let $R \subseteq U \times U$ be a binary relation. Dilation and erosion are key operations used in mathematical morphology and, although we use a different notation, the following definition can be found in [15, p. 61]. The symbols $\oplus$ and $\ominus$ are standard but we depart from the practice of writing erosion in the order $X \ominus R$ in view of Lemma 1.

1 The name BISKT is short for bi-intuitionistic stable *Kt*, where *Kt* is the common notation for the basic tense logic.
2 http://www.mettel-prover.org.