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Observationally-induced lower and upper powerspace constructions ☆

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ABSTRACT

Computational effects can be modelled by observationally-induced algebras, which are algebras whose structure is completely determined by a chosen computational prototype. We show that the category of continuous maps between topological spaces supports a free observationally-induced algebra construction for arbitrary pre-chosen prototype, and give a characterisation of the free algebras as subalgebras of certain powers of these prototypes.
Moreover, we present observationally-induced lower and upper powerspace constructions in the category of topological spaces. Our lower powerspace construction is for all topological spaces given by its non-empty closed subsets equipped with the lower Vietoris topology. Dually, our upper powerdomain construction is for a wide class of topological spaces given by the space of proper open filters of its topology equipped with the upper Vietoris topology. Thus, both constructions generalise the classical construction on continuous dcpos, and unify abstract and concrete characterisations of powerdomains on a broader scale.
Finally, we show that with a small adjustment of the definitions, observationally-induced algebras form the smallest full reflective subcategory of the category of algebras for the corresponding signature, which contains the computational prototype and is complete and closed under isomorphism.

1. Introduction

One of the key problems in denotational semantics is to interpret computational effects, such as user input and output, reading from and writing to memory cells or nondeterministic features, in an abstract mathematical model. The predominant approach to modelling computational effects has been proposed by Moggi [12] in the form of computational monads. Although Moggi's work has provided a unified method of modelling effects and had impact on the design of functional programming languages such as Haskell, it also raised questions as to how to compose monads (for an accurate interplay of effects) and how to obtain the monads in the first place. Therefore, Plotkin and Power [15] have refined Moggi's work, by suggesting to obtain computational monads as free algebra constructions for suitable algebraic theories. The principal idea of this approach is that many computational effects are triggered by algebraic operations, and therefore computational types should be modelled by algebras for these operations. Moreover, the effect triggers obey certain coherence conditions, which can be formulated in terms of algebraic equations that have to be satisfied by the computational types. Thus, in order

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to obtain an appropriate monad for a computational effect, one has to identify the effect triggers with their coherence conditions and translate them into an algebraic theory. The monad is then given by the corresponding free algebra functor. Combining monads is done by combining the operations and equations, potentially adding more equations which express the interplay of the considered effects.

In 2005, Simpson and Schröder [18] have presented a refinement of the Plotkin/Power approach. In this refinement, the effects are still triggered by algebraic operations and the monads are obtained as free algebra functors, but the coherence conditions are defined from a pre-chosen computational prototype instead of being expressed by an equational theory. The computational prototype, which is an algebra for the signature given by the effect triggers, acts as a type defining computational observations, similarly to Sierpinski space defining observable properties for datatypes in Smyth's topological model for denotational semantics [23]. Simpson and Schröder have applied their approach to define an observationally-induced probabilistic powerspace construction in the category of continuous maps between topological spaces.

Retrospectively, one can argue that the classical powerdomains for modelling nondeterminism, introduced by Plotkin [13] and Smyth [21], gave rise to the first computational monads in the sense of Moggi [12]. Originally, these powerdomains were introduced as certain families of subsets, equipped with an appropriate order. Such an explicit characterisation has certain advantages over an abstract free algebra construction, as it allows a better handling of the computational types. Only later the explicit characterisation has been shown to coincide with a free algebra construction on the category of continuous dcpos [1]. The powerdomain characterisations have been generalised by Smyth [22] to fit into his topological framework for denotations semantics, and Schalk [16] has investigated how these constructions can be obtained via free algebra constructions. However, her result was that explicit and abstract constructions coincide only on restricted classes of topological spaces. For a coincidence on a broader scale one has to widen the notion of algebraic theories.

A similar story is true for the probabilistic powerdomain of Jones and Plotkin [9,10]. They have shown that this construction on continuous dcpos can be characterised abstractly as a free algebra construction for convex spaces, and concretely by the set of continuous (sub)probability valuations. Outside the world of continuous dcpos it is known that these constructions differ, as investigated thoroughly by Heckmann [7]. However, Simpson and Schröder [18,20] could show that their abstract observationally-induced probabilistic powerspace construction is for all topological spaces given by the set of continuous (sub)probability valuations.

In this paper we continue and generalise the work of Simpson and Schröder [18] in two ways. First, we show that in the category of continuous maps between topological spaces the observationally-induced construction can be applied for arbitrary pre-chosen computational prototypes. We also show that these constructions inherit desired properties of the pre-defined observation algebra, and characterise it as a subspace of a power of the prototype. Secondly, we investigate observationally-induced lower and upper powerspace constructions on topological spaces. Our lower powerspace construction yields for all topological spaces X the space of non-empty closed subsets of X topologised by the lower Vietoris topology. Our upper powerspace construction yields for a wide class of topological spaces X, namely those that satisfy the so-called Wilker condition, the space of proper open filters of $\mathcal{O}(X)$ equipped with the upper Vietoris topology. We also show that this characterisation does not hold for all topological spaces, by presenting a necessary criterion for it. Finally, we show that with a slight change in the definition of observationally-induced algebras, namely by transferring it completely into the category of algebras, one can show that the observationally-induced algebras form a full reflective subcategory of the category of algebras for the signature given by the effect triggers. In fact, they form the smallest full reflective subcategory which contains the prototype and is complete and closed under isomorphism.

The work at hand should be considered a matured version of our MFPS-conference contribution [3]. The main new results are the characterisation of a necessary criterion for the classical construction of the upper powerspace to coincide with the observationally-induced one in Section 4.3 and, especially, the results on reflectivity in Section 5. Furthermore, the proofs for the characterisation of the lower powerspace in Section 3 have been adjusted to be analogous to the proofs for the characterisation of the upper powerspace in Section 4, thus giving a uniform picture. Finally, in the present work we give full proofs for all claims.

The reader is expected to be familiar with the basic notions of category theory [11], topology [23], domain theory [1] and universal algebra [4].

2. Observationally-induced algebras

We begin by recalling the definition of an observationally-induced free algebra construction in the category **Top** of continuous maps between topological spaces following Schröder and Simpson [18]. Subsequently we recall our results of [3], giving full proofs this time. The main results are that observationally-induced free algebras for arbitrary computational prototypes, and that they can be characterised by subalgebras of certain products of the prototype.

2.1. Basic definitions

Let Σ be a *finitary algebraic signature*, i.e. given by a finite set of operation symbols $\{\sigma \in \Sigma\}$ each of which has an arity $|\sigma| \in \mathbb{N}$. Then a (topological) Σ*-algebra* is a tuple $(A, \{\sigma_A\}_{\sigma\in\Sigma})$ (usually abbreviated as $(A, \{\sigma_A\})$) such that A is a topological space and for all $\sigma \in \Sigma$, $\sigma_A : A^{|\sigma|} \to A$ is a continuous map. A *homomorphism* between Σ-algebras $(A, \{\sigma_A\})$ and $(B, \{\sigma_B\})$ is a continuous map $h : A \to B$ for which $h \circ \sigma_A \equiv \sigma_B \circ h^{|\sigma|}$ holds for all $\sigma \in \Sigma$. Topological Σ-algebras and