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Proper connection number of random graphs ☆

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ABSTRACT

A path in an edge-colored graph is called a proper path if no two adjacent edges of the path are colored with the same color. For a connected graph G, the proper connection number $pc(G)$ of G is defined as the minimum number of colors needed to color its edges, so that every pair of distinct vertices of G is connected by at least one proper path in G. In this paper, we show that almost all graphs have the proper connection number 2. More precisely, let $G(n, p)$ denote the Erdös–Rényi random graph model, in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independent from other pairs. We prove that for sufficiently large n, $pc(G(n, p)) \le 2$ if $p \ge \frac{\log n+\alpha(n)}{n}$, where $\alpha(n) \to \infty$.

1. Introduction

All graphs in this paper are undirected, finite and simple. We follow [4] for graph theoretical notation and terminology not defined here. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \to \{1, 2, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. A path of G is called a *rainbow path* if no two edges on the path have the same color. The graph G is called *rainbow connected* if for any two vertices of G there is a rainbow path of G connecting them. An edge-coloring of a connected graph is called a *rainbow connecting coloring* if it makes the graph rainbow connected. For a connected graph G, the *rainbow connection number* $rc(G)$ of G is the smallest number of colors that are needed in order to make G rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [7] in 2008. The interested readers can see [14,13] for a survey on this topic.

Motivated by the rainbow coloring and proper coloring of graphs, Andrews et al. [1] introduced the concept of proper-path coloring. Let G be a nontrivial connected graph with an edge-coloring. A path in G is called a *proper path* if no two adjacent edges of the path are colored with the same color. An edge-coloring of a connected graph G is a *proper-path coloring* if every pair of distinct vertices of G are connected by a proper path in G. For a connected graph G, the minimum number of colors that are needed to produce a proper-path coloring of G is called the *proper connection number* of G, denoted by $pc(G)$. From the definition, it follows that $1 \le pc(G) \le min\{rc(G), \chi'(G)\} \le m$, where $\chi'(G)$ is the chromatic index of G and m is the number of edges of G. And it is easy to check that $pc(G) = 1$ if and only if $G = K_n$, and $pc(G) = m$ if and only if $G = K_{1,m}$. For more details we refer to [1,5].

The study on rainbow connectivity of random graphs has attracted the interest of many researchers, see [6,11,12]. It is worth investigating the proper connection number of random graphs, which is the purpose of this paper. The most frequently occurring probability model of random graphs is the Erdös–Rényi random graph model $G(n, p)$ [9]. The model $G(n, p)$ consists of all graphs with n vertices in which the edges are chosen independently and with probability p. We say

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an event $\mathcal{A}$ happens *with high probability* if the probability that it happens approaches 1 as $n \to \infty$, i.e., $Pr[\mathcal{A}] = 1 - o_n(1)$. Sometimes, we say *w.h.p.* for short. We will always assume that n is the variable that tends to infinity.

Let G and H be two graphs on n vertices. A property P is said to be *monotone* if whenever $G \subseteq H$ and G satisfies P, H also satisfies P. For any property P of graphs and any positive integer n, define $Prob(P, n)$ to be the ratio of the number of graphs with n labeled vertices having P over the total number of graphs with these vertices. If $Prob(P, n)$ approaches 1 as n tends to infinity, then we say that almost all graphs have the property P. Similarly, for a fixed integer r, we say that almost all r-regular graphs have the property P if the ratio of the number of r-regular graphs with n labeled vertices having P over the total number of r-regular graphs with these vertices approaches 1 as n tends to infinity.

There are many results in the literature asserting that almost all graphs have some property. Here we list some of them, which are related to our study on the proper connection number of random graphs.

Theorem 1.1. *(See [3].) Almost all graphs are connected with diameter 2.*

Theorem 1.2. *(See [3].) For every nonnegative integer k, almost all graphs are k-connected.*

Theorem 1.3. *(See [16].) For fixed integer $r \geq 3$, almost all r-regular graphs are Hamiltonian.*

In [5], Borozan et al. got the following result.

Theorem 1.4. *If the diameter of graph G is 2 and G is 2-connected, then $pc(G) = 2$.*

The authors in [1] proved the following result.

Theorem 1.5. *If G is not complete and has a Hamiltonian path, then $pc(G) = 2$.*

From Theorem 1.1 and Theorem 1.2 and the formula that $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$, it is easy to derive that almost all graphs are 2-connected with diameter 2. Hence, by Theorem 1.4, we have

Theorem 1.6. *Almost all graphs have the proper connection number 2.*

Even if we concentrate on regular graphs, from Theorem 1.3 and Theorem 1.5, we also have the following result.

Theorem 1.7. *For fixed integer $r \geq 3$, almost all r-regular graphs have the proper connection number 2.*

Next, we study the value of the proper connection number of $G(n, p)$, when p belongs to different ranges. The following theorem is a classical result on the connectedness of a random graph.

Theorem 1.8. *(See [9].) Let $p = (\log n + a)/n$. Then*

$$Pr[G(n,p) \text{ is connected}] \to \begin{cases} e^{-e^{-a}} & \text{if } |a| = O(1), \\ 0 & \text{if } a \to -\infty, \\ 1 & \text{if } a \to +\infty. \end{cases}$$

Since the concept of proper-path coloring makes sense only for connected graphs, we only study random graphs $G(n, p)$ which are w.h.p. connected. Our main result is as follows.

Theorem 1.9. *For sufficiently large n, $pc(G(n, p)) \leq 2$ if $p \geq \frac{\log n + \alpha(n)}{n}$, where $\alpha(n) \to \infty$.*

For a graph property P, a function $p(n)$ is called a threshold function of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ w.h.p. satisfies P; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ w.h.p. does not satisfy P.

From Theorem 1.8 and Theorem 1.9, we can obtain that the threshold for $pc(G(n, p)) = 2$ is equal to the threshold for $G(n, p)$ to be connected. We will prove Theorem 1.9 in Section 2.