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On structure and representations of cyclic automata

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ABSTRACT

In this paper we study structure and representations of cyclic automata. Corresponding to Green's equivalences in semigroup theory, we introduce three binary relations say $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on cyclic automata. An automaton is said to be strict if $\mathcal{L}$ is an equivalence on the set of states. Some properties of these relations are established for giving characterizations of three subclasses of strict automata. Also, we provide representations of strict automata by representing the states as vectors and describing the state transitions in terms of matrix operations. These results generalize and extend Ito's representations of strongly connected automata.

1. Introduction

The study of structural and algebraic automata theory was initiated in the 1960s. There are two essential problems in this field, the computation of functions preserving the operations (also known as homomorphisms) of finite automata and the construction of congruences on finite automata. They are important for the decomposition of complex automata and for the specification of the relationships between monoids and automata.

Z. Bavel [3] studied homomorphisms of finite automata systematically. He shows that every automaton can be decomposed into some primaries (also known as maximal cyclic subautomata), and then he characterizes the homomorphisms of an automaton by characterizing the homomorphisms of the primaries. Hence, many statements concerning homomorphisms may be reduced to corresponding statements concerning primaries and many proofs may be simplified with the use of primaries. Much literature has been devoted to characterizations of homomorphisms and isomorphisms of various automata. We refer the reader to [1,2] and [4–13] for further information.

In the 1970s, M. Ito [15–18] promoted the development of the algebraic automata theory. He defines a partial operation $+$ on a group with zero G^0 and constructs a matrix system over $(G^0, +, \cdot)$. The usual matrix multiplication is a closure on this system, although the sum of two elements in G^0 does not always make sense. In addition, some properties of

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vector spaces are still valid. Based on these properties, Ito provides a representation of a strongly connected automaton by representing the states as vectors (in a vector system constructed from the automorphism group) and describing the state transitions in terms of matrix operations. Using this representation, he constructs all the finite strongly connected automata (up to isomorphism) whose automorphism groups are isomorphic to a given G. Thus, he specifies the relationships between finite groups and finite strongly connected automata. However, it is still unclear whether a similar relationship holds between finite monoids and general finite automata.

Ito's thoughts of representations on strongly connected automata seem to come from the representation theory in algebra. Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces and describing its algebraic operations in terms of matrix addition and matrix multiplication. Representation theory is a powerful problem-solving techniques, since it reduces problems in abstract algebra to problems in linear algebra, a subject that is well understood.

Following Bavel and Ito, we study in this paper the structures and representations of cyclic automata by investigating equivalences on them. We list in Section 2 some notions, notation and basic results as preliminaries. In Section 3 we introduce and study three binary relations, say $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$, on the set of states of a cyclic automaton. Some properties on these relations are established. Then we focus on so called strict automata, a subclass of cyclic automata, in which $\mathcal{L}$ is an equivalence. It is shown that an automaton is strongly connected if and only if it is strict and its monoid of endomorphisms is a group. In Sections 4 and 5 we characterize, using the relations $\mathcal{R}$ and $\mathcal{H}$, the strict automata whose monoids of endomorphisms are semilattices and Clifford monoids, respectively. As a generalization of group-matrix type automata (cf. [19]), we introduce and study monoid-matrix type automata in Section 6. Firstly, define a partial operation $+$ on a monoid with zero M^0 and construct a matrix system over $(M^0, +, \cdot)$. Then we show that the usual matrix multiplication and some properties of vector spaces are still valid. It is proved that if M is a finite monoid and $\mathbf{M} = (\widehat{M_n}, X, \delta_\Theta)$ a monoid-matrix type automaton of order n, then M can be embedded into the endomorphism monoid $E(\mathbf{M})$, where $\widehat{M_n}$ is the set of vectors in an n-dimensional vector space constructed from M and Θ is a mapping from X into $(M \cup \{0\})^{n\times n}$ such that $\delta_\Theta(v, x) = v\Theta(x)$ for every $v \in \widehat{M_n}$ and $x \in X$. Further, we provide in Section 7 representations of strict automata which in [24] is used for specifying the relationships between finite Clifford monoids and finite strict automata whose monoids of endomorphisms are Clifford monoids.

2. Preliminaries

An *automaton* $\mathbf{A} = (A, X, \delta)$ consists of the following data:

(i) A is a finite nonempty *set of states*;
(ii) X is a finite nonempty input *alphabet*;
(iii) δ is a function, called the *state transition function*, from $A \times X$ into A.

Let X^* be the set of all finite words over X. Note that X^* forms, with respect to concatenation, the free monoid generated by X in which the empty word ε is the identity element. The state transition function can be extended to a function from $A \times X^*$ to A by

(i) $\delta(a, \varepsilon) = a$ for any $a \in A$;
(ii) $\delta(a, xu) = \delta(\delta(a, x), u)$ for any $a \in A, x \in X$ and $u \in X^*$.

Then for every $w \in X^*$, the mapping

$$\delta_w : A \to A, a \mapsto \delta(a, w),$$

is well-defined.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton. For any $a \in A$, let $\langle a\rangle$ denote the set $\{\delta(a, w) \mid w \in X^*\}$ and let $\mathbf{A}(a) = (\langle a\rangle, X, \delta_{\langle a\rangle \times X})$, where $\delta_{\langle a\rangle\times X}$ is the restriction of δ on the set $\langle a\rangle \times X$, denote the subautomaton of $\mathbf{A}$ generated by a. A state a in A is called a *generator* (cf. [20]) of $\mathbf{A}$ if $\mathbf{A}(a) = \mathbf{A}$. The set of all generators of $\mathbf{A}$ is denoted by $Gen(\mathbf{A})$. An automaton $\mathbf{A}$ is said to be *cyclic* if $Gen(\mathbf{A}) \neq \emptyset$. In particular, $\mathbf{A}$ is said to be *strongly connected* if $Gen(\mathbf{A}) = A$. That is to say, for any pair of states $a, b \in A$, there exists a word $w \in X^*$ such that $\delta(a, w) = b$. In addition, the order $\leq$ on A, defined by

$$a \leq b \Leftrightarrow \langle a\rangle \subseteq \langle b\rangle,$$

is a partial order.

We say an equivalence $\mathcal{T}$ on the set of states A is a *congruence* on $\mathbf{A}$ if $(a, b) \in \mathcal{T}$ implies $(\delta(a, w), \delta(b, w)) \in \mathcal{T}$ for any $a, b \in A$ and any $w \in X^*$.

Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, X, \gamma)$ be automata. A mapping f from A into B is called a *homomorphism* from $\mathbf{A}$ into $\mathbf{B}$ if $f(\delta(a, x)) = \gamma(f(a), x)$ holds for any $a \in A$ and $x \in X$. If a homomorphism f is bijective, then f is called *an isomorphism*. If there exists an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic* to each other and this is denoted by $\mathbf{A} \cong \mathbf{B}$. Moreover, a homomorphism (an isomorphism) from $\mathbf{A}$ into itself is called an *endomorphism* (an *automorphism*) *of* $\mathbf{A}$. It is clear that the set $E(\mathbf{A})(G(\mathbf{A}))$ of all endomorphisms (automorphisms) of $\mathbf{A}$ forms a monoid (group) with respect to the