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Analyzing ultimate positivity for solvable systems

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A B S T R A C T

The positivity problem is a foundational decision problem. It asks whether a dynamical system would keep the observing expression (over its states) positive. It has a derivative—the ultimate positivity problem, which allows that the observing expression is non-positive within a bounded time interval. For the two problems, most existing results are established on discrete-time dynamical systems, specifically on linear recurrence sequences. In this paper, however, we study the ultimate positivity problem for a class of continuous-time dynamical systems, called solvable systems. They subsume linear systems. For the general solvable system, we present a sufficient condition for inferring ultimate positivity. The validity of the condition can be algorithmically checked. Once it is valid, we can further find the time threshold, after which the observing expression would be always positive. On the other hand, we show that the ultimate positivity problem is decidable for some special classes of solvable systems, such as linear systems of dimension up to five.

1. Introduction

Skolem–Pisot problem is a famous decision problem in theoretical computer science. Generally speaking, it asks whether a (discrete-time or continuous-time) dynamical system would make the observing expression $f(\mathbf{x})$ (over its states $\mathbf{x}$) reach zero. The **positivity** problem is its complement, which asks whether a dynamical system would keep the observing expression $f(\mathbf{x})$ positive. The two decision problems have received increasing interests in the last decade, because they are closely related to many newly-emerging fields, such as *program verification* [27,30], *probabilistic model checking* [2], and *quantum automata* [7]. The two decision problems are significant but intrinsically intractable. Hence most existing progress is established either on approximate decision problems or on restricted system models.

In the past, Skolem–Pisot problem was usually interpreted over linear recurrence sequences (LRSs for short) $\langle x(\tau)\rangle_{\tau\in\mathbb{Z}^+}$—a class of discrete-time dynamical systems. In this setting, the observing expression $f(x)$ is simply the state x of the given LRS. A profound result was Skolem–Mahler–Lech theorem [25,16,14], stating that the set of zero states $\{\tau \in \mathbb{Z}^+ \mid x(\tau) = 0\}$ in a LRS is the union of finitely many periodic sets and a finite set. Later Hansel reproved this classic theorem by p-adic method [10]. In 1980s, Mignotte et al. and Vereshchagin independently showed that Skolem–Pisot problem is decidable for LRSs of order up to four [17,28]. A comprehensive survey on these works was given in [9], in which the authors further attacked the decidability for fifth order LRSs. But the proof of the concluding result (see Proposition 4.7 of [9]) seemed to have a serious gap (as reported in [19]). Thus Skolem–Pisot problem is still open for LRSs of order five or higher.

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In the meantime, progress on the positivity problem of LRSs is fairly slow, since the pilot study in 1970s [23]. In 2006, Halava et al. showed the decidability of the positivity for second order LRSs [8]. Three years later, Laohakosol and Tangsupphathawat showed that for third order LRSs [13]. The decidability for high order LRSs was no doubt very hard. A breakthrough was the recent paper by Ouaknine and Worrell [19]. The authors applied powerful number-theoretic results to decide the positivity for LRSs of order up to five, and indicated that the decidability for sixth order LRSs would make major progress in Diophantine approximation. Besides, they also concerned the positivity for simple LRSs, in which characteristic polynomials have no repeated root, and concerned the **ultimate positivity** problem, which allows only a finite number of non-positive states. They decided the positivity for simple LRSs of order up to nine [18], and decided the ultimate positivity for simple LRSs in polynomial time [20]. For complicated LRSs with variable coefficients, the positivity problem becomes more intractable. The known decidability results are established for some special third order LRSs [12].

Recently, Bell et al. first considered the continuous Skolem–Pisot problem [3], in which Skolem–Pisot problem was interpreted over linear systems—a class of continuous-time dynamical systems. They showed that Skolem–Pisot problem is decidable for linear systems of dimension up to two and for some special classes of linear systems. But, in general, the continuous Skolem–Pisot problem has been proven to be NP-hard in [3] (c.f. the discrete version was reported in [4]), and its decidability is still unknown at the time of writing. Naturally the positivity and the ultimate positivity problems are worthy of concern for continuous-time dynamical systems. As far as we know, however, existing methods and results are rare on them. An indirect way for inferring positivity is the method of **barriers** generation. It first computes a superset of reachable states $\mathbf{x}$ of the given dynamical system, which is defined by a barrier; and then asserts the positivity holds if the whole superset satisfies the desired property $f(\mathbf{x}) > 0$. For instance, utilizing convex optimization techniques, Prajna constructed barriers between the reachable states and the unsafe states for nonlinear dynamical systems [21]. However, the generated barriers are usually polynomial, whereas characterizing the set of reachable states of most dynamical systems (including linear systems) needs more expressibility than semi-algebraic sets. Hence the barrier-based proof is not always sharp enough to infer positivity for continuous-time dynamical systems.

In this paper, we study the ultimate positivity problem for a class of continuous-time dynamical systems, called solvable systems (a superset of linear systems). We first give the formal description of solvable systems, whose solutions are shown to have a closed form. Then we group all terms in the solution into different growth classes. The coefficients of terms in the greatest growth class build up the so-called leading component, whose sign would dominate the signs of other components when the time variable is sufficiently large, i.e. the time variable is beyond certain time threshold T. On the basis of it, we present some sufficient conditions for inferring and refuting ultimate positivity, which are dependent on the infimum of the leading component. The validity of these conditions can be checked by algebraic algorithms. Hence these conditions are computable. Once the ultimate positivity is inferred by our condition, the time threshold T can further be found. We reduce this task to the upper bound of real roots of multi-exponential polynomials (a class of univariate real functions to be defined later on).

The above method works for the general solvable system, and it is enough to infer or refute ultimate positivity for most instances. However it fails to establish the decidability result. For decidability, we have to restrict our focus to some special classes of solvable systems. Technically, we propose restrictions on the structures of components in the solution of the solvable system, saying the leading component having at most three distinct arguments. Then it is ensured by [18] that the leading component reaches its infimum only at finitely many periodic sets and a finite set. For the former, we are required to further consider other components to yield an entire result. For the latter, after certain computable time threshold, we can bound the leading component away from its infimum by an inverse polynomial. Thus we can prove that for simple solvable systems (to be explained later on), the sign of the leading component would dominate those of other components when the time variable is sufficiently large. Thereby we successfully decide the ultimate positivity problem for those special classes of solvable systems, such as linear systems of dimension up to five.

Organization In Section 2 we review some basic notions and results from number theory. In Section 3 we introduce solvable systems and the positivity problems. Then we present computable conditions for inferring and refuting ultimate positivity in Section 4, and show the decidability for some special classes in Section 5. Finally we draw a conclusion in Section 6.

2. Preliminaries

Here we briefly review some number-theoretic notions and results for analyzing the positivity problems afterwards.

Definition 2.1. The number α is *algebraic*, denoted by $\alpha \in \mathbb{A}$, if there exists a nonzero irreducible polynomial $\rho(x) \in \mathbb{Z}[x]$ (named the *minimal polynomial*) such that $\rho(\alpha) = 0$; otherwise it is *transcendental*.

Every algebraic number is entirely determined by its minimal polynomial confined in a finite area in the complex plane. Let $\mathbb{R}_{\text{alg}}$ be short for the real algebraic number field $\mathbb{R} \cap \mathbb{A}$. For instance, an eigenvalue λ of a rational matrix is algebraic, and both its real part $\mu = \Re(\lambda)$ and its imaginary part $\nu = \Im(\lambda)$ are real algebraic.

Definition 2.2. The numbers $\nu_1, \ldots, \nu_n$ are *linearly independent* if no linear relation $\sum_{i=1}^{n} z_i \nu_i = 0$ with integer coefficients z_i, not all zero, holds among them; otherwise they are *linearly dependent*.