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The number and degree distribution of spanning trees in the Tower of Hanoi graph

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ABSTRACT

The number of spanning trees of a graph is an important invariant related to topological and dynamic properties of the graph, such as its reliability, communication aspects, synchronization, and so on. However, the practical enumeration of spanning trees and the study of their properties remain a challenge, particularly for large networks. In this paper, we study the number and degree distribution of the spanning trees in the Hanoi graph. We first establish recursion relations between the number of spanning trees and other spanning subgraphs of the Hanoi graph, from which we find an exact analytical expression for the number of spanning trees of the n-disc Hanoi graph. This result allows the calculation of the spanning tree entropy which is then compared with those for other graphs with the same average degree. Then, we introduce a vertex labeling which allows to find, for each vertex of the graph, its degree distribution among all possible spanning trees.

1. Introduction

The problem of finding the number of spanning trees of a finite graph is a relevant and long standing question. It has been considered in different areas of mathematics [1], physics [2], and computer science [3], since its introduction by Kirchhoff in 1847 [4]. This graph invariant is a parameter that characterizes the reliability of a network [5–7] and is related to its optimal synchronization [8] and the study of random walks [9]. It is also of interest in theoretical chemistry, see for example [10]. The number of spanning trees of a graph can be computed, as shown in many basic texts on graph theory [11], from Kirchhoff's matrix-tree theorem [12] and it is given by the product of all nonzero eigenvalues of the Laplacian matrix of the graph. Although this result can be applied to any graph, the calculation of the number of spanning trees from the matrix theorem is analytically and computationally demanding, in particular for large networks. Not surprisingly, recent work has been devoted to finding alternative methods to produce closed-form expressions for the number of spanning trees for particular graphs such as grid graphs [13], lattices [14–17], the small-world Farey graph [18–20], the Sierpiński gasket [21,22], self-similar lattices [23,24], etc.

Most of the previous work focused on counting spanning trees on various graphs [1]. However, the number of spanning trees is an integrated, coarse characteristic of a graph. Once the number of spanning trees is determined, the next step is to explore and understand the geometrical structure of spanning trees. In this context, it is of great interest to compute the probability distribution of different coordination numbers at a given vertex among all the spanning trees [25], which

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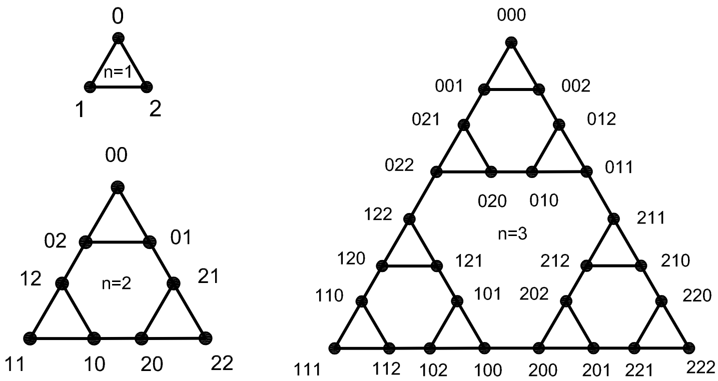

Fig. 1. Hanoi graphs H_1, H_2 and H_3.

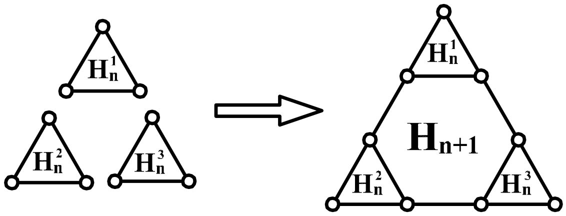

Fig. 2. Construction rules for the Hanoi graph. H_{n+1} is obtained by connecting three graphs H_n labeled here by H_n^1, H_n^2 and H_n^3.

encodes useful information about the role the vertex plays in the whole network. Due to the computational complexity of the calculation, this geometrical feature of spanning trees has been studied only for very few graphs, such as the $\mathbb{Z}^d$ lattice [26], the square lattice [27], and the Sierpiński graph [28]. It is non-trivial to study this geometrical structure for other graphs.

In this paper, we study the number and structure of spanning trees of the Hanoi graph. This graph, which is also known as the Tower of Hanoi graph [29], comes from the well known Tower of Hanoi puzzle, as the graph is associated to the allowed moves in this puzzle. There exist an abundant literature on the properties of the Hanoi graph, which includes the study of shortest paths, average distance, planarity, Hamiltonian walks, group of symmetries, average eccentricity, to name a few, see [29] and references therein. In [24], Teufl and Wagner obtained the number of spanning trees of different self-similar lattices, including the Hanoi graph. Here, based on the self-similarity of the Hanoi graph, we enumerate its spanning trees and compute for each vertex of the graph its degree distribution among all spanning trees.

2. The Hanoi graph

The Hanoi graph is derived from the Tower of Hanoi puzzle with n discs [29]. We can consider each legal distribution of the n discs on the three peg, a state, as a vertex of the Hanoi graph, and an edge is defined if one state can be transformed into another by moving one disc. If we label the three pegs 0, 1 and 2, any legal distribution of the n discs can be written as the vector/sequence $\alpha_1\alpha_2\ldots\alpha_n$ where α_i $(1 \leq i \leq n)$ gives the location of the $(n+1-i)$th largest disc. We will denote as H_n the Hanoi graph of n discs. Fig. 1 shows H_1, H_2 and H_3.

Note that H_{n+1} $(n \geq 1)$ can be obtained from three copies of H_n joined by three edges, each one connecting a pair of vertices from two different replicas of H_n, as shown in Fig. 2. From the construction rule, we find that the number of vertices or order of H_n is 3^n while the number of edges is $\frac{3}{2}(3^n - 1)$.

In the next section will make use of this recursive construction to find the number of spanning trees of H_n at any iteration step n.

3. The number of spanning trees in H_n

If we denote by V_n and E_n the number of vertices and edges of H_n, then a spanning subgraph of H_n is a graph with the same vertex set as H_n and a number of edges E'_n such that $E'_n < E_n$. A spanning tree of H_n is a spanning subgraph that is a tree and thus $E'_n = V_n - 1$.

In this section we calculate the number of spanning trees of the Hanoi graph H_n. We adapt the decimation method described in [30–32], which has also been successfully used to find the number of spanning trees of the Sierpiński gasket [22], the Apollonian network [33], and some fractal lattices [16]. This decimation method is in fact the standard renormalization group approach [34] in statistical physics, which applies to many enumeration problems on self-similar graphs [35]. We