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Dynamic monopolies with randomized starting configuration

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ABSTRACT

Properties of systems with majority voting rules have been extensively studied. In this work we focus on the randomized case, where the system is initialized by random initial set of seeds. Our main aim is to find an asymptotic estimate for sampling probability, so that the initial set of seeds is (not) almost surely a dynamic monopoly. After presenting some trivial examples, we present extensive results for toroidal mesh and random 4-regular graph under simple majority scenario.

1. Introduction

The idea of *majority voting* is commonly used to resolve many problems related to achieving consistency among different parts of distributed computation. For example, majority voting is used to preserve data consistency when updating copies of the same data. It is also quite common to use majority voting to resolve inconsistencies in distributed database management. Majority based systems were also successfully used by Agur to examine the plasticity and precision of the immune response in [2].

The model for the system is as follows. Let $G = (V, E)$ be a simple undirected graph of size n. Every vertex has its color, which is either black or white. The color represents the state of the node (for example black = faulty, white = not faulty). By S we shall denote the set of black vertices at the beginning of the process. These vertices are also called *seeds*. By evolution of such a system (or, the *coloring process*) we shall mean the synchronous process where in each step each vertex adjusts its color according to colors of its neighbors and its internal contamination and decontamination rules. (De)contamination rules determine under what configuration of its neighbors the white (black) vertex turns black (white). In this work we focus on the case when there is no decontamination rule, and the contamination rule is the simple majority rule. This means that the white vertex turns black if at least half of its neighbors are black and there is no possibility for black vertex to turn white. The set of seeds S is called *dynamic monopoly*, (or *dynamo* for short) if the corresponding coloring process leads to monochromatic black graph. For more rigorous definition, see Section 2.1.

Significant attention was paid to this model and many interesting results were obtained. Probably the most basic (but certainly not trivial) question asks for determining the minimal cardinality of a dynamo on a fixed graph G [9,10]. Another interesting parameter of a dynamo besides its cardinality is the time that is needed for contamination to spread. This was analyzed for the first time in the literature in [9,10]. Several works are related to more advanced topics such as decontamination of the system by external agents [12]. Finally, [14] defines the terminology of immune subgraphs and asks how the immune subgraph of a certain graph looks like.

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All these tasks were solved for small classes of graphs. Close attention was given to the ring and its modifications [9], as well as to the torus and its modifications [10] (toroidal mesh, torus cordalis, torus serpentinus). Many results were also obtained for hypercube and binary tree. For toroidal mesh the size of minimum dynamo is $n+1$ under simple and $\frac{n^2}{3}\cdot(1+o(1))$ under strict majority scenario. This means that the initial frequency of black vertices is $\Theta(1/n)$ and $1/3+o(1)$ respectively. The following is one of our contributions: if the initial contamination was chosen uniformly at random, then the initial frequency of black vertices that leads to dynamo (w.h.p.) would be $\Theta(1/\ln(n))$ for simple and $1-o(1)$ for strong majority scenario. (The first statement will be proved later, the second statement is trivial.)

Other possible questions ask about the minimal cardinality of a dynamo on arbitrary graphs. It was proved that the minimal dynamo on general directed graph has at most $0.727|V|$ ($0.7732|V|$) vertices under simple (strict) majority scenario [8]. Recently, this result was significantly improved to $|V|/2$ ($2|V|/3$) [1]. In the case of general undirected graphs the minimal dynamo consists of at most $\lceil V/2\rceil$ ($\lfloor V/2\rfloor+1$) vertices.

There are some interesting results about such systems even in the "most general" case, where the vertex is contaminated if at least fraction α of its neighbors are black [6,13,16]. These works answer the question, under what conditions at least some fraction δ of all vertices is turned black.

Finally, we mention several works that we find to be most related to this paper. For tree-like graphs and randomly chosen set of initial black vertices, Gleeson and Cahalane [13] gave an exact formula for the expected fraction of black vertices at the end. In [6,7], the authors gave their estimate for a minimal number of black vertices needed for re-coloring of fraction δ of all vertices on the Erdős–Rényi random graph. In [7], the authors proved that using random graph $G(n,p)$ as an underlying graph, the size of minimum dynamo is $\Omega(n/\ln(n))$; the upper bound is not discussed. (Note that the authors use a random graph, but the initial contamination is not chosen randomly.) For more thorough survey, see [11,15].

Motivation for studying a coloring process induced by a random set of seeds is quite straightforward. For example, considering vertices as computing nodes, each node can fail with probability p, independently of failure of other nodes. Although this looks like a very common scenario, there are only few results for systems with random initial coloring. Given graph G we consider random initial set of seeds $S^p = S^p(G)$ as the set containing any vertex of G with probability p (independently of other vertices). Naturally, for every fixed graph G this gives us some probability that the random set of seeds is a dynamo. Determining these probabilities analytically is difficult (if not impossible) and moreover, it can be done numerically with sufficient accuracy. Therefore, we shall try to obtain asymptotic results of the form: assuming 4-regular graph with n vertices, random set of seeds $S^{0.12}$ is dynamo with high probability (w.h.p.). On the other hand, $S^{0.10}$ is not a dynamo (w.h.p.).

In many situations, it is trivial to determine the minimal value of p, so that S^p is dynamo (w.h.p). For example, assuming the Erdős–Rényi random graph $G(n,p')$ for fixed p', S^p almost surely (a.s.) forms a dynamo if $p > 1/2+\varepsilon$ (where $\varepsilon > 0$ is an arbitrarily small constant) and a.s. does not form a dynamo if $p < 1/2-\varepsilon$ (this follows from Chernoff bounds and the Markov inequality). If we take the same model and allow p' to be dependent on n, but do not allow p' to be significantly decreasing, the same results can be easily derived. Finally, if p' decreases significantly with n (for example $p = c/n$), the graph contains some isolated vertices (with probability tending to some $\gamma > 0$). Therefore, we cannot say that S^p is w.h.p. a dynamo, until p is not quite close to 1 (so close that all isolated vertices are a.s. seeds). Another trivial example is the toroidal mesh under strict majority contamination rule (that is a white vertex turns black only if it has at least three black neighbors). In this case, any square of size 2×2 forms an immune subgraph (that is, such a subgraph that turns black only if some of its vertices are seeds). Therefore, the sampling probability that would form dynamic monopoly must prevent all such squares from being colored entirely white. Once again, this can be done only for p very close to 1.

However, the motivation presented in the previous text leads to considering p to be a fault probability of a node in the network (or something similar). Therefore, we can assume $1-p$ not to be very small. This makes the cases where $p\to 1$ unrealistic.

In this work we shall examine dynamic monopolies with random initial conditions on two types of underlying graphs. In Section 2, we focus on the toroidal mesh. The results are quite surprising — S^p containing only a fraction $o(1)$ of all the vertices a.s. form a dynamo. To be more concrete, we shall show that there exist constants α,β such that if $p > \alpha/\ln(n)$, then S^p a.s. is a dynamo. Similarly, when $p < \beta/\ln(n)$, then S^p a.s. is not a dynamo. At the end of the section, we present our attempt to solve the problem numerically and we show our "measured" α and β. As was said earlier, to form a dynamo on a random graph $G(n,p')$ we need p to be close to 1/2. This makes the results for the toroidal mesh even more interesting. The natural question arises, whether the reason that such low values of p are needed to form a dynamo comes from specific topology of toroidal mesh, or whether it is just implied by the fact, that the degree of the vertices is constant and low. This is the motivation for Section 3, where we investigate the same questions for random 4-regular graphs, that is the graphs with all the vertices having degree 4. We show that if $p\geq 0.12$, then the random initial coloring a.s. forms a dynamo. Similarly, if $p\leq 0.10$, then the random initial coloring a.s. does not form a dynamo.

2. Toroidal mesh

2.1. Preliminaries

By *toroidal mesh of size n* we shall mean an undirected graph $G = G(V,E)$ consisting of n^2 vertices labeled as $V[i,j]$ for $0\leq i,j < n$. The set of edges consists of all pairs $(V[i,j], V[(i+1) \mod n, j])$ and $(V[i,j], V[i,(j+1) \mod n])$ for all