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Aspects of predicative algebraic set theory, II: Realizability

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Dedicated to Jean-Yves Girard on the occasion of his 60th birthday.

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ABSTRACT

One of the main goals of this paper is to give a construction of realizability models for predicative constructive set theories in a predicative metatheory. We will use the methods of algebraic set theory, in particular the results on exact completion from van den Berg and Moerdijk (2008) [5]. Thus, the principal results of our paper are concerned with the construction of an extension of a category with small maps by a category of assemblies, again equipped with a class of maps, and to show that this extension construction preserves those axioms for a class of maps necessary to produce models of the relevant set theories in the exact completion of this category of assemblies.

1. Introduction

This paper is the second in a series on the relation between algebraic set theory [20] and predicative formal systems. The purpose of the present paper is to show how realizability models of constructive set theories fit into the framework of algebraic set theory. It can be read independently from the first part [5]; however, we recommend that readers of this paper read the introduction to [5], where the general methods and goals of algebraic set theory are explained in more detail.

To motivate our methods, let us recall the construction of Hyland's effective topos $\mathcal{E}\!f\!f$ [18]. The objects of this category are pairs $(X, =)$, where $=$ is a subset of $\mathbb{N} \times X \times X$ satisfying certain conditions. If we write $n \Vdash x = y$ in case the triple (n, x, y) belongs to this subset, then these conditions can be formulated by requiring the existence of natural numbers s and t such that

$$s \Vdash x = x' \to x' = x$$
$$t \Vdash x = x' \wedge x' = x'' \to x = x''.$$

These conditions have to be read in the way usual in realizability [36]. So the first says that for any natural number n satisfying $n \Vdash x = x'$, the expression $s(n)$ should be defined and be such that $s(n) \Vdash x' = x$.[1] And the second stipulates that for any pair of natural numbers n and m with $n \Vdash x = x'$ and $m \Vdash x' = x''$, the expression $t(\langle n, m\rangle)$ is defined and is such that $t(\langle n, m\rangle) \Vdash x = x''$.

The arrows $[F]$ between two such objects $(X, =)$ and $(Y, =)$ are equivalence classes of subsets F of $\mathbb{N} \times X \times Y$ satisfying certain conditions. Writing $n \Vdash Fxy$ for $(n, x, y) \in F$, one requires the existence of realizers for statements of the form

$$Fxy \wedge x = x' \wedge y = y' \to Fx'y'$$

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1 For any two natural numbers n, m, the Kleene application of n to m will be written $n(m)$, even when it is undefined. When it is defined, this will be indicated by $n(m)\downarrow$. We also assume that some recursive pairing operation has been fixed, with the associated projections being recursive. The pairing of two natural numbers n and m will be denoted by $\langle n, m\rangle$. Every natural number n will code a pair, with its first and second projection denoted by n_0 and n_1, respectively.

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$$Fxy \to x = x \wedge y = y$$
$$Fxy \wedge Fxy' \to y = y'$$
$$x = x \to \exists y\, Fxy.$$

Two such subsets F and G represent the same arrow $[F] = [G]$ iff they are extensionally equal in the sense that

$$Fxy \leftrightarrow Gxy$$

is realized.

As shown by Hyland, the logical properties of this topos $\mathcal{E}\!f\!f$ are quite remarkable. Its first-order arithmetic coincides with the realizability interpretation of Kleene [21]. The interpretation of the higher types in $\mathcal{E}\!f\!f$ is given by **HEO**, the hereditary effective operations. Its higher-order arithmetic is captured by realizability in the manner of Kreisel and Troelstra [35], so as to validate the uniformity principle:

$$\forall X \in \mathcal{P}\mathbb{N}\, \exists n \in \mathbb{N}\, \phi(X, n) \to \exists n \in \mathbb{N}\, \forall X \in \mathcal{P}\mathbb{N}\, \phi(X, n).$$

The topos $\mathcal{E}\!f\!f$ is one in an entire family of *realizability toposes* defined over arbitrary partial combinatory algebras (or more general structures modeling computation). The relation between these toposes has been not been completely clarified, although much interesting work has already been done in this direction [31,18,24,8,16,15] (for an overview, see [30]). The construction of the topos $\mathcal{E}\!f\!f$ and its variants can be internalised in an arbitrary topos (we will always assume our toposes to have a natural numbers object). This means in particular that one can construct toposes by iterating (alternating) constructions of sheaf and realizability toposes to obtain interesting models for higher-order intuitionistic arithmetic **HHA**. An example of this phenomenon is the modified realizability topos, which occurs as a closed subtopos of a realizability topos constructed inside a presheaf topos [29].

The purpose of this series of papers is to show that these results are not only valid for toposes as models of **HHA**, but also for certain types of categories equipped with a class of small maps suitable for constructing models of constructive set theories like **IZF** and **CZF**. In the first paper of this series [5], we have axiomatised this type of categories, and refer to them as "predicative categories with small maps". For the convenience of the reader their precise definition is recalled in Appendix B, while the axioms of the set theories **IZF** and **CZF** are reviewed in Appendix A.

A basic result from [5] is the following:

Theorem 1. *Every predicative category with small maps $(\mathcal{E}, \mathcal{S})$ contains a model (V, ϵ) of a weak set theory (to be precise,* **CZF** *without Subset collection). Moreover,*

(i) (V, ϵ) *is a model of* **IZF***, whenever the class $\mathcal{S}$ satisfies the axioms* **(M)** *and* **(PS)***.*
(ii) (V, ϵ) *is a model of* **CZF***, whenever the class $\mathcal{S}$ satisfies* **(F)***.*[2]

To show that realizability models fit into this picture, we prove that predicative categories with small maps are closed under internal realizability, in the same way that toposes are. More precisely, relative to a given predicative category with small maps $(\mathcal{E}, \mathcal{S})$, we construct a "predicative realizability category" $(\mathcal{E}\!f\!f_{\mathcal{E}}, \mathcal{S}_{\mathcal{E}})$. The main result of this paper will then be:

Theorem 2. *If $(\mathcal{E}, \mathcal{S})$ is a predicative category with small maps, then so is $(\mathcal{E}\!f\!f_{\mathcal{E}}, \mathcal{S}_{\mathcal{E}})$. Moreover, if $(\mathcal{E}, \mathcal{S})$ satisfies one of the axioms* **(M), (F)** *or* **(PS)***, then so does $(\mathcal{E}\!f\!f_{\mathcal{E}}, \mathcal{S}_{\mathcal{E}})$.*

We show this for the pca $\mathbb{N}$ together with the Kleene application, but we expect that this result can be proved in the same way, when $\mathbb{N}$ is replaced by a pca $\mathcal{A}$ in $\mathcal{E}$, provided that both $\mathcal{A}$ and the domain of the application function $\{(a, b) \in \mathcal{A}^2 : a \cdot b \downarrow\}$ are small. The proof of the theorem above is technically rather involved, in particular in the case of the additional properties needed to ensure that the model of set theory satisfies the precise axioms of **IZF** and **CZF**. However, once this work is out of the way, one can apply the construction to many different predicative categories with small maps, and show that familiar realizability models of set theory (and some unfamiliar ones) appear in this way.

One of the most basic examples is that where $\mathcal{E}$ is the classical category of sets, and $\mathcal{S}$ is the class of maps between sets whose fibres are all bounded in size by some inaccessible cardinal. The construction underlying Theorem 2 then produces Hyland's effective topos $\mathcal{E}\!f\!f$, together with the class of small maps defined in [20], which in [23] was shown to lead to the Friedman–McCarty model of **IZF** [13,26] (we will reprove this in Section 5).

An important point we wish to emphasise is that one can prove all the model's salient properties without constructing it explicitly, using its universal properties instead. We explain this point in more detail. A predicative category with small maps consists of a category $\mathcal{E}$ and a class of maps $\mathcal{S}$ in it, the intuition being that the objects and morphisms of $\mathcal{E}$ are classes and class morphisms, and the morphisms in $\mathcal{S}$ are those that have small (i.e., set-sized) fibres. For such predicative categories with small maps, one can prove that the small subobjects functor is representable. This means that there is a *power class object* $\mathcal{P}_s(X)$ which classifies the small subobjects of X, in the sense that maps $B \longrightarrow \mathcal{P}_s(X)$ correspond bijectively to jointly monic diagrams

$$B \longleftarrow U \longrightarrow X$$

[2] The precise formulations of the axioms **(M)**, **(PS)** and **(F)** can be found in Appendix B as well.