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Musings around the geometry of interaction, and coherence

Jean Goubault-Larrecq

LSV, ENS Cachan, CNRS, INRIA Saclay, 61 avenue du président Wilson, 95230 Cachan, France

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ABSTRACT

We introduce the Danos–Régnier category $\mathcal{DR}(M)$ of a linear inverse monoid M, as a categorical description of geometries of interaction (GOI) inspired from the weight algebra. The natural setting for GOI is that of a so-called weakly Cantorian linear inverse monoid, in which case $\mathcal{DR}(M)$ is a kind of symmetrized version of the classical Abramsky–Haghverdi–Scott construction of a weak linear category from a GOI situation. It is well-known that GOI is perfectly suited to describe the multiplicative fragment of linear logic, and indeed $\mathcal{DR}(M)$ will be a $*$-autonomous category in this case. It is also well-known that the categorical interpretation of the other linear connectives conflicts with GOI interpretations. We make this precise, and show that $\mathcal{DR}(M)$ has no terminal object, no Cartesian product of any two objects, and no exponential—whatever M is, unless M is trivial. However, a form of coherence completion of $\mathcal{DR}(M)$ à la Hu–Joyal (which for additives resembles a layered approach à la Hughes–van Glabbeek), provides a model of full classical linear logic, as soon as M is weakly Cantorian. One finally notes that Girard's notion of *coherence* is pervasive, and instrumental in every aspect of this work.

1. Introduction

There are by now several families of models for (classical) linear logic. One is the category of *coherence spaces* [17]. Another is given by game models, e.g. [4]. Contrarily to what one might expect, geometry of interaction, in whatever form [14–16,19] does not yield models of linear logic. Now, by *model* of linear logic we are rather demanding, and mean a denotational, in fact a *categorical* model. The definition of categorical models of linear logic took some time to emerge, and is certainly posterior to geometry of interaction. We shall consider linear categories [7], LNL categories [6], Lafont and new-Lafont categories [27]. It is remarkable that coherence spaces form a model in all these senses, but most proposals based on games or geometry of interaction do not. The point is subtle: e.g., Baillot et al. [4] show that AJM games are a model of MELL proof nets (i.e., without the additives) without box erasure steps. Some more recent game semantics, such as Melliès' asynchronous games [28], do provide a categorical model of linear logic.

In a sense, there are categorical models of a domain-theoretic style, but only a few coming from the interaction world, and none from the geometry of interaction. This paper bridges the gap. Our main contribution is a categorical model of full classical linear logic, including multiplicative, exponential and additive connectives, based on ideas from geometry of interaction – specifically from Danos and Régnier [11,10] – and also using the notion of *coherence completion* [21]. So we import from both interaction and domain theory. *Coherence* plays a fundamental role in both.

A word on the organization of this paper. First, we feel that some intuition about the roots of this work should be brought forward, and we devote Section 2 to this. We introduce the concept of a *linear inverse semigroup M* in Section 3, and show

E-mail address: goubault@lsv.ens-cachan.fr.

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in Section 4 how any such M gives rise to a category $\mathcal{DR}(M)$, which we call the *Danos–Régnier category* of M. We shall also see that, provided M is *weakly Cantorian*, $\mathcal{DR}(M)$ is compact-closed. In particular, it is a model of the multiplicative fragment MLL of linear logic. The purpose of Section 5 is to compare this construction to the $\mathcal{G}$ construction of Abramsky et al. [1], a.k.a. Joyal et al.'s *Int* construction [23], the most prominent categorical interpretation of geometry of interaction. On our way to obtaining a categorical model of the whole of linear logic, we shall then trip on a serious difficulty: we shall show in Section 6 and Section 7 that there is no way to interpret *any* form of additive or exponential connective in $\mathcal{DR}(M)$, whatever M. I.e., changing the languages of paths won't help. Nonetheless, we show in Section 8 that a slight modification of Hu and Joyal's *coherence completion* [21] builds a Lafont category out of any $*$-autonomous category, i.e., a model of full classical linear logic out of any model of just MLL . . ., and this is exactly what $\mathcal{DR}(M)$ provides, no more, no less.

Another word on related work. We shall discuss related work throughout the paper, notably the construction of compact-closed categories from traced monoidal categories [1,23] in Section 5, and coherence completions [21] in Section 8. The idea of considering inverse monoids is credited to Yves Legrandgérard by Danos and Régnier [11]. As far as the impossibility results mentioned in Sections 6 and 7 are concerned, it is well-known that trying to add specific new equations between geometry of interaction tokens, aimed at enforcing some categorical identities, resulted in inconsistencies. Our impossibility results are much stronger: we show that *no* change in the underlying inverse monoid M can result in the creation of *any* instance of any missing categorical feature (additive, exponential).

2. Motivation

I came to study inverse monoids following Danos et al. [10], where weights from the so-called dynamic algebra arise from an inverse monoid with some added structure (the *bar*, which captures the reduction process). However, my actual initial goal was to try and understand how one may describe Böhm-like trees of λ-terms up to β- or $\beta\eta$-equivalence, not as trees, but as collections of paths through these trees. (A goal I have not reached yet.)

Let us see what this means for trees. By tree we mean some form of infinite first order term: each node t is labeled by a function symbol f of some arity $n \in \mathbb{N}$, and has n successors $t_1, \ldots, t_n$; we then agree to write t as $f(t_1, \ldots, t_n)$. We call Σ the given signature, i.e., the set of all function symbols, together with their respective arities. We write $f/n \in \Sigma$ to state that f is in Σ, with arity n. With each such f/n in Σ, we associate n distinct letters $f_1, \ldots, f_n$. (We need to adjust this when $n = 0$, in all rigor.) This yields the path alphabet $|\mathcal{A}| = \bigcup_{f/n \in \Sigma}\{f_1, \ldots, f_n\}$. Its elements are the path letters, and a *path* is any finite sequence of path letters. Traveling down a tree along any route from the root yields a path in the obvious way. E.g., the tree $f(g(t_1, t_2), t_3)$ has (at least) the paths ϵ (the empty path), f_1, f_1g_1, f_1g_2, f_2.

Going from a tree to its set of paths is easy. Recovering a tree from a given set of paths is harder. First, not every set of paths arises from some tree, e.g., $\{f_1, g_1\}$. The key point to enable this reconstruction process is *coherence*. This was invented under a different name by Harrison and Havel [20]. Define an equivalence relation $\equiv$ on the path alphabet by $f_i \equiv g_j$ iff $f = g$. Now let $\frown$ be the relation on paths such that $w \frown w'$ iff, for any strict common prefix w_0 of w and w', writing w as w_0aw_1 and w' as $w_0a'w'_1$ with $a, a' \in |\mathcal{A}|$, then $a \equiv a'$; $\frown$ is reflexive and symmetric, though in general not transitive. When $w \frown w'$, we say that w and w' are *coherent*, and a *clique* is any set of pairwise coherent paths. Clearly, any set of paths of a given tree is a clique. In general, a space $X = (|X|, \frown)$ where $\frown$ is a reflexive and symmetric relation on $|X|$ is a *coherence space* [17]. So there is a coherence space of paths, $(|\mathcal{A}|^*, \frown)$; this was explored by Reddy [31, Section 5.2]. Coherence spaces form the basis of an elegant semantics of the λ-calculus, and in fact of all of linear logic [17].

Let us refine. Let $\leq$ be the prefix ordering on paths. Then $w \leq w'$ and $w' \frown w''$ implies $w \frown w''$: $(|\mathcal{A}|^*, \leq, \frown)$ is a bit more than a coherence space, it is an *event structure*, i.e., a space $X = (|X|, \leq, \frown)$ where $\leq$ is a partial ordering and $\frown$ is a reflexive and symmetric relation on $|X|$ such that $w \leq w'$ and $w' \frown w''$ implies $w \frown w''$. Then the set of paths in a tree is a *down-closed* clique, and conversely any down-closed clique is the set of paths of a unique tree (except that functions f/n may have less than n subtrees).

Event structures are a fundamental model of concurrency [29], where, instead of using $\frown$, a binary irreflexive and symmetric relation # called *conflict* is used, such that $w \leq w'$ and $w\#w''$ implies $w'\#w''$. (We have also ignored the axiom of so-called finite causes here.) This is equivalent: take coherence $\frown$ as negation of conflict #. The relationship between order $\leq$ and coherence $\frown$ is explained, and generalized to so-called bistructures, by Curien et al. [9].

In the case of λ-terms, as opposed to infinite first-order terms, there is an extra difficulty in identifying terms with certain cliques of paths: λ-terms reduce to other λ-terms, and we would like to define a notion of paths through λ-terms that is *invariant* under $\beta\eta$-equivalence. The result will be a way to compute paths through the Böhm tree of t by just computing paths through t itself—*without* reducing t. This is exactly what geometry of interaction is about. Girard's execution formula aims at being such an invariant. Our view is that such an invariant should be a denotational (categorical) model of λ-calculus, and in fact of linear logic proofs.

3. Linear inverse semigroups

Such a calculus of paths for MLL terms is lurking around in [11,10], based on the notion of a (bar) inverse monoid. The quantity that remains invariant through reduction is the set of all weights of paths through a proof net. But this cannot