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Some results about the chaotic behavior of cellular automata☆

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Abstract

We study the behavior of cellular automata (CA for short) in the Cantor, Besicovitch and Weyl topologies. We solve an open problem about the existence of transitive CA in the Besicovitch topology. The proof of this result has some interest of its own since it is obtained by using Kolmogorov complexity. To our knowledge it is the first result about discrete dynamical systems obtained using Kolmogorov complexity. We also prove that in the Besicovitch topology every CA has either a unique periodic point (thus a fixed point) or an uncountable set of periodic points. This result underlines the fact that CA have a great degree of stability; it may be considered a further step towards the understanding of CA periodic behavior.

Moreover, we prove that in the Besicovitch topology there is a special set of configurations, the set of Toeplitz configurations, that plays a role similar to that of spatially periodic configurations in the Cantor topology, that is, it is dense and has a central role in the study of surjectivity and injectivity. Finally, it is shown that the set of spatially quasi-periodic configurations is not dense in the Weyl topology.

Keywords: Cellular automata; Deterministic chaos; Unconventional topologies; Transitivity; Kolmogorov complexity

1. Introduction

In the last 20 years cellular automata (CA for short) received a growing attention as formal models for complex systems with applications in almost every scientific domain. They consist in an infinite lattice of identical finite automata. Each automaton updates its state according to a *local rule* on the basis of its present state and those of a finite set of neighboring automata. The states of all automata are updated synchronously. A *configuration* is a snapshot of all the states of all automata in the lattice.

The simplicity of the definition of this model is in contrast with the (perhaps only apparent) variety of dynamical behaviors, most of which are not completely understood yet.

The dynamical behavior of CA has been studied mainly in the context of discrete dynamical systems by endowing the set of configurations with the classical Cantor topology (i.e. the product topology when the set of states of the

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automaton is equipped with the discrete topology). Deterministic chaos is one of the most appealing features of dynamical behaviors. It is far from being completely understood. Among CA, one can find various interesting examples of this kind of behavior.

The problem is that in the Cantor topology the *shift* map is chaotic according to Devaney's popular definition of chaos [14]: a dynamical system is chaotic in this sense if it is transitive and regular (i.e. there exists a dense set of periodic points).

The shift map is a very simple CA; it just shifts the content of configurations one step to the left. It is a model of chaoticity in Information Theory and Ergodic Theory but, viewed as a CA, its chaoticity is somewhat counter-intuitive (see [9,3] for a discussion on this topic). In fact, the chaoticity of the shift is more an artefact of the Cantor topology, or of the Devaney definition of chaos, than a consequence of the intrinsic complexity of the automaton [15,10]. In [9], in order to avoid what they felt to be biasses of the Cantor topology (in the context of chaotic behavior), the authors proposed to replace it by the Besicovitch topology. In [18,3], the authors proved that this topology better links the classical notion of sensitivity to initials conditions with the intuitive notion of chaotic behavior. The Besicovitch topology deserves to be studied for at least one other reason: in statistical mechanics, the intuitive idea that two spatial configurations are close to each other translates easily into the assumption that their Besicovitch distance is small.

In the Cantor topology we study strong transitivity, a property that is strictly stronger than transitivity. We prove that strongly transitive CA are not injective. Recalling that, for CA, injectivity is equal to reversibility, this result underlines the fact that in strongly transitive systems there is a real global information loss. This means moreover that when substituting strong transitivity to transitivity in Devaney's definition of chaos the shift ceases to be chaotic (but in this case the shift on one-sided sequences is still chaotic). We also prove that all positively expansive CA are strongly transitive, one more step in the hierarchy of chaos [13].

In the case of the Besicovitch topology, we address three topics: stability, surjectivity and transitivity. Concerning stability, we prove that all CA have either a unique fixed point or an uncountable set of periodic points. This result points out that CA have a great degree of local stability. This fact is further stressed by the extension to the Besicovitch topology of a result in [5]: surjective CA having a blocking word are regular. Here we give a proof which does not use measure theory. This strengthens the conjecture that all surjective CA are regular [5,12,6].

Spatially periodic configurations, or rather their equivalence classes, are not dense in the Besicovitch topology. We study the relation between surjectivity and the wider set of Toeplitz configurations. We prove that in the Besicovitch topology it is dense, and with respect to surjectivity plays a role similar to that of the set of periodic configurations in the Cantor topology. Moreover, we prove that, exactly like in the Cantor case, injectivity implies surjectivity both for the global map and for its restriction to Toeplitz configurations.

Afterwards, we negatively answer the question of existence of transitive CA in the Besicovitch topology; this was called a challenging open problem in [23]. This result has deep implications for CA dynamics. First, it states that CA cannot change arbitrarily the density of differences between two configurations during their evolutions. In its turn this fact implies that the information contained in configurations cannot spread too much during evolutions. Second, it opens the quest for new, more appropriate, properties for describing the "complex" behavior of CA dynamics. Some very interesting proposals along this line of thought may be found in [24]. Third, the proof technique is of some interest of its own. We make use of the theory of Kolmogorov complexity and the famous incompressibility method [22] to prove a result of pure topological dynamics.

The paper ends with some results about similar problems in the Weyl topology. We first prove that the set of quasi-periodic configurations (containing Toeplitz sequences) is not dense in this topology. Then we address the question of finding an interesting dense set which could replace Toeplitz configurations in the Weyl setting, and prove that the set of non-generic configurations with respect to the uniform Bernoulli measure is dense.

2. Cellular automata

Formally, a CA is a quadruple $\langle d, S, N, \lambda\rangle$. The integer d is the *dimension* of the CA and controls how the cells of the lattice are indexed. Indeed, indexes of cells take values in $\mathbb{Z}^d$. The symbol S ($|S| < \infty$) is the finite set of states of cells and $\lambda\colon\ S^N \to S$ is the *local rule* which updates the state of a cell on the basis of a (finite) *neighborhood* $N \subset \mathbb{Z}^d$.

A *configuration* c is a function from $\mathbb{Z}^d$ to S and may be viewed as a snapshot of the content of each cell in the lattice. Denote by X the set $S^{\mathbb{Z}^d}$ of all configurations.