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H-Coloring dichotomy revisited

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Abstract

The H-Coloring problem can be expressed as a particular case of the constraint satisfaction problem (CSP) whose computational complexity has been intensively studied under various approaches in the last several years. We show that the dichotomy theorem proved by Hell and Nešetřil [On the complexity of H-coloring, J. Combin. Theory Ser. B 48 (1990) 92–110] for the complexity of the H-Coloring problem for undirected graphs can be obtained using general methods for studying CSP, and that the criterion distinguishing the tractable cases of the H-Coloring problem agrees with that conjectured in [A.A. Bulatov, P.G. Jeavons, A.A. Krokhin, Constraint satisfaction problems and finite algebras, in: Proc. 27th Internat. Colloq. on Automata, Languages and Programming—ICALP'00, Lecture Notes in Computer Science, Vol. 1853, Springer, Berlin, 2000, pp. 272–282] for the complexity of the general CSP.

Keywords: Graph colouring; Complexity; Constraint satisfaction; Algebraic approach

1. Introduction

The computational complexity of the H-Coloring problem and related problems such as List H-Coloring, Counting H-Coloring, Restrictive H-Coloring has been intensively studied during the last two decades (for a comprehensive survey see [10,12]). One of the most prominent results achieved in this research direction is the *dichotomy theorem* for undirected graphs [11] that establishes that the H-Coloring problem is solvable in polynomial time (we shall call such problems *tractable*) if and only if H has a loop or is a bipartite graph; otherwise the problem is NP-complete. We call this result a dichotomy theorem, because it leaves only two possibilities for an undirected graph: to give rise either to a tractable problem or to an NP-complete problem. Notice that if $\mathrm{P} \neq \mathrm{NP}$ then there are infinitely many pairwise distinct complexity classes between P and NP [17]. In this paper, we assume $\mathrm{P} \neq \mathrm{NP}$.

The H-Coloring problem can be considered within a more general framework, the constraint satisfaction problem (CSP, for brevity). In the CSP associated with a finite relational structure $\mathcal{H}$ (we denote it by $\mathrm{CSP}(\mathcal{H})$), the question is whether there exists a homomorphism of a given finite relational structure to $\mathcal{H}$. Thus, the H-Coloring problem is a particular case of the CSP in which the involved relational structures are graphs.

One of the major research problems in studying the CSP is so-called *classification problem* aiming to distinguish those relational structures which give rise to tractable CSPs from those which do not. Several approaches to tackle the classification problem using methods from logic, algebra, game theory and database theory have been developed

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recently (see e.g. [5,7,9,15,16]), that has made it possible to achieve substantial progress [1–3,6,8,13,14,23]. This allowed Feder and Vardi [9] to conjecture that the dichotomy *tractable—NP-complete* holds for the general CSP.

The algebraic approach that has proved to be very successful uses methods and results from universal algebra, and provides a deep insight into the structure of the CSP. In particular, algebraic concepts make it possible to conjecture a plausible criterion distinguishing tractable and NP-complete CSPs [5]. (For necessary definitions and results see Section 2.) Almost all known results on the complexity of the CSP have been shown to agree with this criterion. The H-Coloring dichotomy theorem is one of the few remaining results for which it is not yet proved.

In this paper we reprove the dichotomy theorem from [11] using algebraic methods. We pursue two main goals. The first one is to illustrate how these methods can be used to obtain results about graph homomorphisms. In order to do that, in Section 2, we give an outline of the algebraic approach attempting to translate as much as possible algebraic terminology and results in graph theory terms. The second goal we achieve is to show that the criterion for the tractability of undirected H-Coloring problems is a particular case of the algebraic criterion from [5]. Theorem 1 establishes this fact. As a by-product we also get a shorter and simpler proof of the result of [11].

2. Definitions and techniques

2.1. Constraint satisfaction problem

The CSP can be equivalently defined in several ways. It is convenient for us to define the CSP as the Homomorphism problem. A *vocabulary* is a finite set of relational symbols $R_1, \ldots, R_n$ each of which has a fixed arity. A *relational structure* over the vocabulary $R_1, \ldots, R_n$ is a tuple $\mathcal{H} = (H; R_1^{\mathcal{H}}, \ldots, R_n^{\mathcal{H}})$ such that H is a non-empty set, called the *universe* of $\mathcal{H}$, and each $R_i^{\mathcal{H}}$ is a relation on H having the same arity as the symbol R_i. (We shall omit the index $\mathcal{H}$ whenever it does not lead to a confusion.) Let $\mathcal{G}, \mathcal{H}$ be relational structures over the same vocabulary $R_1, \ldots, R_n$. A *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a mapping $\varphi : G \to H$ from the universe G of $\mathcal{G}$ to the universe H of $\mathcal{H}$ such that, for every relation $R^{\mathcal{G}}$ of $\mathcal{G}$ and every tuple $(\mathbf{a}[1], \ldots, \mathbf{a}[m]) \in R^{\mathcal{G}}$, we have $(\varphi(\mathbf{a}[1]), \ldots, \varphi(\mathbf{a}[m])) \in R^{\mathcal{H}}$.

Let $\mathcal{H}$ be a relational structure over a vocabulary $R_1, \ldots, R_n$. In the *constraint satisfaction problem associated with* $\mathcal{H}$, denoted CSP($\mathcal{H}$), the question is, given a structure $\mathcal{G}$ over the same vocabulary, whether there exists a homomorphism from $\mathcal{G}$ to $\mathcal{H}$.

A (directed) graph $\mathcal{H} = (V; E)$ can be treated as a relational structure with one binary relation. Thus, the $\mathcal{H}$-Coloring problem is equivalent to CSP($\mathcal{H}$).

A relational structure $\mathcal{H}$ is said to be *tractable* if CSP($\mathcal{H}$) is tractable; it is said to be *NP-complete* if CSP($\mathcal{H}$) is NP-complete. Often it is convenient to call a set of relations Γ on H tractable if any relational structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ such that $R_1, \ldots, R_n \in \Gamma$ is tractable. The set Γ is said to be NP-complete if, for certain $R_1, \ldots, R_n \in \Gamma$, the structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ is NP-complete.

We use the standard correspondence between relations and predicates defined on the same set. In particular, we use the same symbol for a relation and for the corresponding predicate.

In [13,15], it has been shown that adding to a relational structure relations derived using certain rules does not change the complexity of the corresponding CSP. Let Γ be a set of relations. The set of relations derivable from Γ is defined to be the set of relations definable by *primitive positive formulas* (*pp-formulas* for short) involving the relations of Γ and the equality relation:

Definition 1. For any set of relations Γ over H, the set $\langle\Gamma\rangle$ consists of all relations that can be expressed using
1. relations from Γ, together with the binary equality relation on H (denoted $=_H$),
2. conjunction, and
3. existential quantification.

We say that a relation R is *definable* in a relational structure $\mathcal{H} = (H; R_1, \ldots, R_n)$ if $R \in \langle\{R_1, \ldots, R_n\}\rangle$.

Example 1 (*Multiplication of binary relations*). Let R_1, R_2 be binary relations on a set H. Then the relation $R_1 \circ R_2$, the product of R_1, R_2, is the relation definable by the pp-formula $(R_1 \circ R_2)(x, y) = \exists z(R_1(x, z) \wedge R_2(z, y))$. We use R^n to denote the nth power of R, the relation $\underbrace{R \circ \ldots \circ R}_{n \text{ times}}$.