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Group coloring is Π_2^P-complete

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Abstract

The group chromatic number of a graph G is the smallest integer k such that for every Abelian group A of order at least k, every orientation of G and every edge-labeling $\varphi : E(G) \to A$, there exists a vertex-coloring $c : V(G) \to A$ with $c(v) - c(u) \neq \varphi(uv)$ for each oriented edge uv of G. We show that the decision problem whether a given graph has group chromatic number at most k is Π_2^P-complete for each integer $k \geqslant 3$.

Keywords: Group coloring; Group connectivity; Nowhere-zero flows; Π_2^P-Completeness

1. Introduction

Group colorings of graphs have been introduced by Jaeger et al. [4]. For plane graphs, this concept is dual (in the usual sense) to group connectivity that generalizes the intensively studied concept of nowhere-zero flows in graphs. For an Abelian group A, a graph G is said to be A-*colorable* if for every orientation of G and for every *edge-labeling* $\varphi : E(G) \to A$, there is a vertex coloring $c : V(G) \to A$ such that $c(w) - c(v) \neq \varphi(vw)$ for each oriented edge $vw \in E(G)$. Note that the choice of an orientation of edges of G is not essential since reversing the orientation of the edge e can be replaced by changing the forbidden difference $\varphi(e)$ to $-\varphi(e)$. A plane graph is A-colorable iff its dual is A-connected [4]. We remark that it is unknown whether the property of being A-colorable (A-connected) depends on the structure or only on the order of the group A (like in the case of nowhere-zero flows [13]). The least number $\chi_g(G)$ such that G is A-colorable for each Abelian group A of order at least $\chi_g(G)$ is the *group chromatic number* of G.

Group colorings have attracted a lot of attention from the combinatorial point of view. Some theorems for ordinary colorings, e.g., Brooks’ theorem, can be adopted to group colorings [7], some cannot. However, the concept of group coloring is closer to the concept of list-coloring [6]: the group chromatic number of a graph G with the average degree d is at least $\Omega(d/\log d)$. Hence, as in the concept of list coloring, the group chromatic number of graphs with large average degree is large. Lai and Zhang [8] proved that the group chromatic number of planar graphs is bounded by five and the author with Pangrác and Voss [6] constructed a planar graph with the group chromatic number five. This is the same bound as in the case of list-coloring [11]. Planar graphs without 3-cycles and 4-cycles have group chromatic number

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at most three and planar graphs without 4-cycles at most four [6] matching the same bounds for choosability [9,12]. On the other hand, the choice number of planar bipartite graphs is at most three [1] but there is a planar bipartite graph with the group chromatic number equal to four [6].

In this paper, we address the complexity of group coloring. We show that for each Abelian group A of order at least three it is Π_2^P-complete to decide whether a given graph G is A-colorable. The problem to decide whether the group chromatic number of a graph G is at most k is also Π_2^P-complete for each integer $k \geqslant 3$.

2. Definitions and notation

All groups considered in this paper are finite and Abelian. The neutral element is always denoted by 0. Elements of the group are sometimes also referred to as colors. If an orientation and edge-labeling φ of a graph G is given, then a vertex coloring c is *proper* if $c(v) - c(u) \neq \varphi(uv)$ for every oriented edge uv of G. Hence, a graph G is A-colorable if there is a proper coloring for each orientation of it and each edge-labeling $\varphi : E(G) \to A$. An edge uv with $c(v) - c(u) \neq \varphi(uv)$ is said to be *properly* colored. On the other hand, if $c(v) - c(u) = \varphi(uv)$, the colors of u and v *conflict*.

Let us state a simple lemma about group coloring that is used later.

Lemma 1. *Let v be a vertex of a graph G. If G is A-colorable, then for each orientation and each edge-labeling, there exists a proper coloring c of G with $c(v) = 0$.*

We first show that a variant of group coloring where list sizes are not the same for all vertices of a graph is Π_2^P-complete and then we derive the Π_2^P-completeness of the original problem. A *list-size-assigning* function is a mapping $\psi : V(G) \to \{1, \dots, |A|\}$. An *$A$-list-$\psi$-assignment* for a graph G and an Abelian group A is a mapping $L : V(G) \to 2^A$ with $|L(v)| = \psi(v)$ for every vertex $v \in V(G)$. If the group A is clear from the context, the list assignment L is just said to be a *list-ψ-assignment*. The graph G is said to be *A-ψ-choosable* if for each orientation of G, each edge-labeling $\varphi : E(G) \to A$ and each list-ψ-assignment L, there is a vertex coloring $c : V(G) \to A$ such that $c(v) \in L(v)$ for every vertex $v \in L(v)$ and $c(w) - c(v) \neq \varphi(vw)$ for every oriented edge $vw \in E(G)$. We also say that G can be colored from lists L or that it is *A-L-colorable* in such a situation. If ψ is a constant function equal to ℓ, then the list assignment L is said to be *list-ℓ-assignment* and the graph G is said to be *A-ℓ-choosable*. Note that a graph is A-$|A|$-choosable iff it is A-colorable. Group choosability has sometimes a somewhat strange behavior, e.g., a cycle of even length is A-2-choosable for every group A of odd order and it is not A-2-choosable for any group A of even order.

The complexity class Π_2^P is defined to be the class of the complements of problems that can be solved by non-deterministic algorithms running in polynomial time with access to the oracle solving problems from the class NP [2,10]. Clearly, the class Π_2^P is a superclass of NP. If the definition is iterated, one obtains the class Σ_3^P, together with the complementary class Π_3^P, that consists of the problems that can be solved by non-deterministic algorithms running in polynomial time with access to the oracle solving problems from Π_2^P. Similarly, the classes Σ_k^P and Π_k^P are defined [2,10].

Very few natural combinatorial problems are known to be Π_2^P-complete, e.g., the problem to decide whether a graph is 3-choosable [3], or the problem to compute the generalized Ramsey number [2] are Π_2^P-complete. Similarly, as ordinary satisfiability of formulas is the basic NP-complete problem, the decision problem whether a given Π_2-formula is true is the basic Π_2^P-complete problem [10]. A formula Ψ is a *Π_2-3CNF-formula* if it is of the following form:

$$\Psi = \forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n \Psi_0(x_1, \dots, x_m, y_1, \dots, y_n),$$

where Ψ_0 is a 3CNF-formula with variables $x_1, \dots, x_m, y_1, \dots, y_n$, i.e., a formula Ψ_0 is in the conjunctive normal form (CNF) and each clause of it has size exactly three. Formulas Ψ where Ψ_0 is of this restricted form are called *Π_2-3CNF-formulas*. The *size* of a Π_2-3CNF-formula is the number of its clauses. Occurrences of variables in clauses of Ψ_0 are called *literals*. A literal is *positive* if it is of the form x_i, $i = 1, \dots, m$, or y_i, $i = 1, \dots, n$. Similarly, a literal is *negative* if it is of the form $\neg x_i$, $i = 1, \dots, m$, or $\neg y_i$, $i = 1, \dots, n$.